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ABSTRACT

This document outlines part two of the third year course of the newly implemented New York State Mathematics Curriculum. The three-year Sequential Mathematics sequence designed by New York State is a move in the direction of fusing the formerly separate topics found in algebra, geometry, and intermediate algebra/trigonometry, and introducing timely topics which have been absent from the course of study. The guide is based upon the scope and sequence for Sequential Mathematics, Course III, part 2. It consists of plans for lessons, which should be considered suggestions rather than prescriptions. There is an abundance of application problems, provided as a resource for teachers. The activities emphasize problem solving skills, reasoning skills, and critical thinking. Though Course III is written with 11th grade students in mind, it is possible that some students may begin the material earlier or later, or complete it in less or more than 1 year. (PK)

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ED289696

Pilot Edition — 1986

SEQUENTIAL MATHEMATICS

Course III

Part 2

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Alaire Mitchell

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FOREWORD

The Sequential Mathematics Sequence provides New York City students with an integrated course of study for grades nine, ten, and eleven. In a departure from traditional courses of study, lessons interweave algebra, geometry, logic, probability, and mathematical systems. This course extends the integrated approach that begins in kindergarten and continues through grade 8.

In the Sequential Mathematics Sequence, algebra, geometry, and trigonometry remain important components of the curriculum. In addition, other branches of mathematics -- such as probability, logic, and mathematical systems -- are introduced and emphasized as important tools for dealing with present and future technological developments.

Each lesson in Course III begins with a clearly stated aim presented in question form. Performance objectives follow; these provide a specific focus for the teacher. The new vocabulary for the lesson is presented. This is followed by a motivational activity. A full development is presented, with suggested pivotal questions. Applications, solutions to applications, and a summary are also included. In each lesson opportunities have been infused for helping students to develop critical thinking, problem-solving, and reasoning skills. These activities will enable students to become more skilled in higher levels of thinking.

The Sequential Mathematics Sequence has been designed to anticipate students' needs for mathematical skills in an increasingly technologically-oriented culture. We anticipate that the Sequential Mathematics Sequence will serve the individual needs of your students and enable them to acquire the mathematics skills they need in the present and for the future.

Charlotte Frank

Charlotte Frank
Executive Director
Division of Curriculum and Instruction

ACKNOWLEDGMENTS

Sequential Mathematics, Course III, Part II was developed for teachers of Sequential Mathematics as a project of the Mathematics Unit of the Division of Curriculum and Instruction, Charlotte Frank, Executive Director.

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INTRODUCTION

Sequential Mathematics (Course III, Part II) is the third year course of the newly implemented New York State curriculum. This publication from the Division of Curriculum and Instruction is a guide for teachers. It attempts to present the material covered in this course in a way that is most comprehensive and useful to the classroom teacher.

Traditionally, in New York State, mathematics in grades nine, ten, and eleven has consisted of algebra, geometry, and intermediate algebra/trigonometry. Students completing the three-year sequence have been exposed to these topics as relatively independent units. Moreover, students discontinuing their study of mathematics after grade eleven have never been exposed to such timely and practical topics as probability, statistics, and formal logical reasoning. Those terminating their high school mathematics program after grade ten have been given even fewer opportunities for exposure to these topics. The three-year Sequential Mathematics sequence designed by New York State is a move in the direction of fusing these formerly separate topics and introducing those which have been absent from the course of study.

This curriculum guide is based upon the Scope and Sequence for Sequential Mathematics, Course III, Part II. It consists of plans for lessons, which should be considered suggestions rather than prescriptions. Each of these plans stress the development of mathematical concepts, with an emphasis on providing opportunities for the development of problem solving skills, reasoning skills and critical thinking. To help students develop these skills, each plan contains suggested questions to encourage student investigation and thought. The problems presented in the plans are designed to be relevant to students' experiences. There is an abundance of application problems.
These have been provided as a resource for teachers so that they may select appropriate problems for different levels of classes. Some objectives and application problems are marked with an asterisk and may be more appropriate for above-average classes.

Though Course III is written with eleventh grade students in mind, it is possible that some students may begin the material earlier or later, or complete it in less than or more than one year. It is expected that individual teachers will consider their students' abilities and interests in making judgments concerning time allotments, aspects of material to be stressed, or extra material to be presented.

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LESSON: 1

AIM: How do we draw the graph of $y = \sin x$ and $y = \cos x$?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Draw the graphs of $y = \sin x$ and $y = \cos x$ over the domain $-2\pi \le x \le 2\pi$ using intervals of $\frac{\pi}{6}$.
2. Identify the amplitude and period of $y = \sin x$ and $y = \cos x$.
3. Find the quadrants in which sin x is increasing.
4. Find the quadrants in which cos x is increasing.

VOCABULARY: amplitude, period

MOTIVATION:

1. Why is sin x called a trigonometric function?

   Students should state that it is a function because for each value of x there is only one value for y. Furthermore, because it is a function then we should be able to draw the graph of that function.

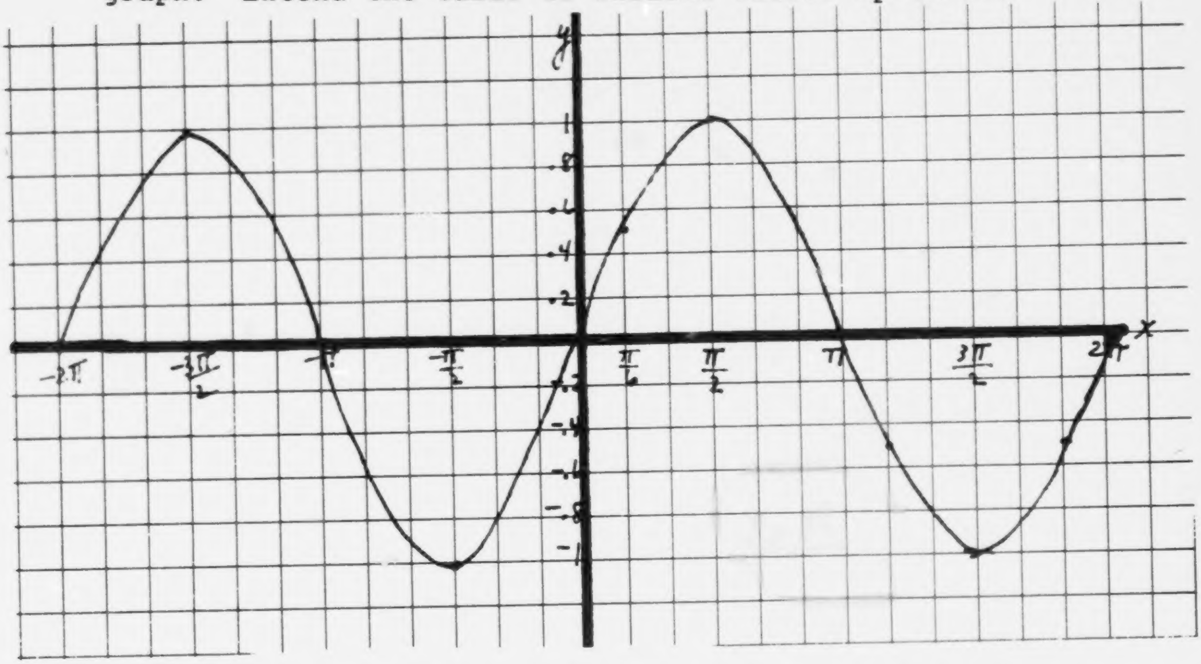
DEVELOPMENT:

1. Explain that the values of x and y on the coordinate axes are real numbers and, therefore, it is necessary to express the values of x in radians rather than in degrees. Ask the class to fill in the table below.

x	y = sin x	y
0		
$\frac{\pi}{6}$		
$\frac{\pi}{3}$		
$\frac{\pi}{2}$		

Show that the values of x are taken in intervals of $\frac{\pi}{6}$ radians or 30°.

After completing the table show the class how to set up the axes. A convenient approach is to allow each value of x to represent $\frac{\pi}{6}$ radians and to let 5 squares on the y-axis represent 1 unit as shown below. Then fill in the graph. Extend the table to include values up to 2π.



Show that the graph proceeds in the opposite direction to -2π and complete that graph.

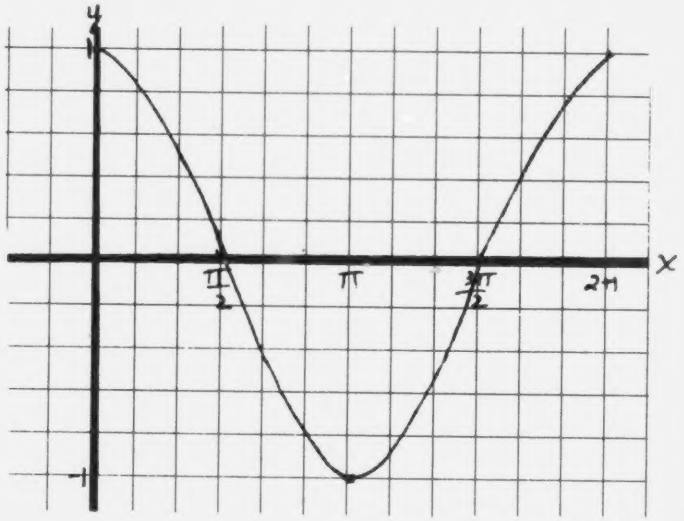
2. Have the class study the graph and reach the following conclusions:
   - a) The maximum height the curve reaches in one. This is called the amplitude. In general, the amplitude of a trigonometric function is the difference between the maximum value and the minimum value divided by 2.
   - b) 1) From x = 0 to x = $\frac{\pi}{2}$, sin x is increasing.
     2) From x = $\frac{\pi}{2}$ to x = π, sin x is decreasing.
     3) From x = π to x = $\frac{3\pi}{2}$, sin x is decreasing.
     4) From x = $\frac{3\pi}{2}$ to x = 2π, sin x is increasing.
   - c) The trigonometric function values of the quadrantal angles of sin x can be read from this graph. In many cases it is easier to use this graph to find the function values of the quadrantal angles.
   - d) After 2π radians the sine curve repeats itself. Define period - as the number of radians needed to complete one cycle. The period for y = sin x is 2π radians.

3. Have the class draw the graph of $y = \cos x$. In this case have the class discover they need not find each individual value of $\cos x$ but can use the values found for $y = \sin x$ as a guide. Discuss the following pertaining to $y = \cos x$ with the class:
   a) Amplitude
   b) Variation
   c) Quadrantal values
   d) Period

   Elicit the graph of $y = \cos x$, the amplitude is 1, the period of the graph is 2π and that

   cos x is decreasing in quadrant I
   cos x is decreasing in quadrant II
   cos x is increasing in quadrant III
   cos x is increasing in quadrant IV

   Also elicit $\cos 0° = 1$, $\cos \frac{\pi}{2} = 0$, $\cos \pi = -1$,
   $\cos \frac{3\pi}{2} = 0$, $\cos 2\pi = 1$



APPLICATIONS:

1. In what quadrants are both sin x and cos x both
   a) increasing? b) decreasing?

2. As x increases from $\frac{\pi}{2}$ to π, what change takes place in
   a) sin x? b) cos x?

3. For what value(s) of x in the interval $-2\pi \leq x \leq 2\pi$ does
   a) $\sin x = 1$ b) $\cos x = 1$ c) $\sin x = -1$
   d) $\cos x = -1$

SOLUTIONS TO APPLICATION PROBLEMS:

1. a) IV b) II

2. a) decreases from 1 to 0

b) decreases from 0 to -1

3. a) $x = \frac{\pi}{2}$, $x = \frac{-3\pi}{2}$

b) $x = 0$, $x = 2\pi$ and $x = -2\pi$

c) $x = \frac{3\pi}{2}$, $x = \frac{-\pi}{2}$

d) $x = \pi$, $x = -\pi$

SUMMARY:

1. What is the meaning of amplitude and period of a function? What is the amplitude and period of $y = \sin x$ and $y = \cos x$?
2. What is the range of a) $y = \sin x$? b) $y = \cos x$?
3. In the interval $0 \leq x \leq 2\pi$, for what values of x does $y = \sin x$ a) reach a maximum? b) reach a minimum? c) cross the x-axis?
4. In the interval $0 \leq x \leq 2\pi$ for what values of x does $y = \cos x$ a) reach a maximum? b) reach a minimum? c) cross the x-axis?

LESSON: 2

AIM: How do we sketch the graphs of $y = a \sin x$ and $y = a \cos x$?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Sketch the graphs of $y = a \sin x$ and $y = a \cos x$ using maximum points, minimum points and points of intersection with the x-axis.
2. State the amplitude and periods of $y = a \sin x$ and $y = a \cos x$.
3. Find the number of points of intersection of $y = a \sin x$ and $y = a \cos x$ when they are drawn on the same set of axes.

MOTIVATION:

Ask the class to find $f(0)$ and $f(\frac{\pi}{2})$ for each of the following functions:

1) $f(x) = \sin x$
2) $f(x) = 4 \sin x$
3) $f(x) = \frac{1}{2} \sin x$

After going over the answers to the above questions ask the following question: "What effect does the "a" have on the graph of the function $y = a \sin x$?"

From the answers to the above problems, students should state that the maximum value of the function becomes "a" and that the amplitude is "a".

DEVELOPMENT:

1. Have the class fill in the table below:

x	$y = 2\cos x$	y
0		
$\frac{\pi}{2}$		
π		
$\frac{3\pi}{2}$		
2π		

From the answers in the sketch the graph of $y = 2\cos x$. Explain that the sketch must contain each of the points in the table but it is not necessary to fill in a table for each problem.

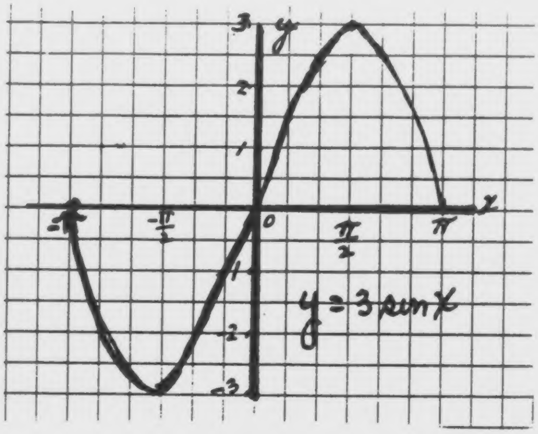
2. Have the class sketch $y = \frac{1}{2}\cos x$ over the interval $0 \le x \le 2\pi$.
3. Sketch $y = -4\cos x$ with the class. Point out that the amplitude is 4.
4. Develop rule - The amplitude of $y = a \sin x$ and $y = a \cos x$ is $|a|$.
5. On the same set of axes sketch $y = 2\sin x$ and $y = 3\cos x$ over the interval $0 \le x \le 2\pi$. Also determine the quadrants in which the graphs intersect.

APPLICATIONS:

1. Sketch the graph of $y = 3\sin x$ over the interval $-\pi \le x \le \pi$.
2. Sketch the graph of $y = -4\cos x$ over the interval $0 \le x \le 2\pi$.
3. Which trigonometric function has the greatest amplitude?
   a) $y = 6\cos x$ b) $y = -10\sin x$ c) $y = \frac{1}{20}\cos x$
4. a) Sketch the graphs of $y = 2\sin x$ and $y = -\cos x$ on the same set of axes over the interval $0 \le x \le 2\pi$.
   b) In what quadrants does $2\sin x = -\cos x$.
5. a) Sketch $y = -2\sin x$ and $y = 2$ in the interval $0 \le x \le 2\pi$.
   b) In how many points does $y = -2\sin x$ and $y = 2$ intersect in the interval 1) $0 \le x \le \pi$
   2) $\pi \le x \le 2\pi$

SOLUTIONS TO APPLICATIONS:

1)



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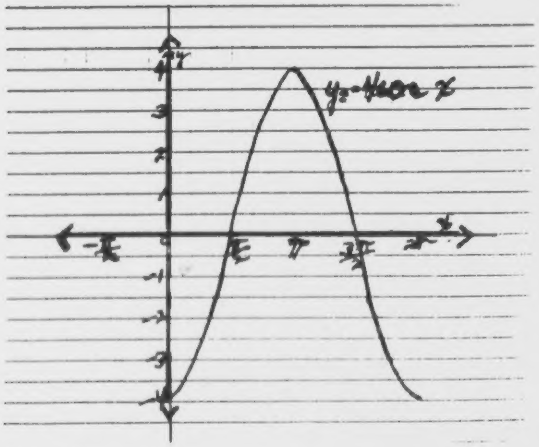
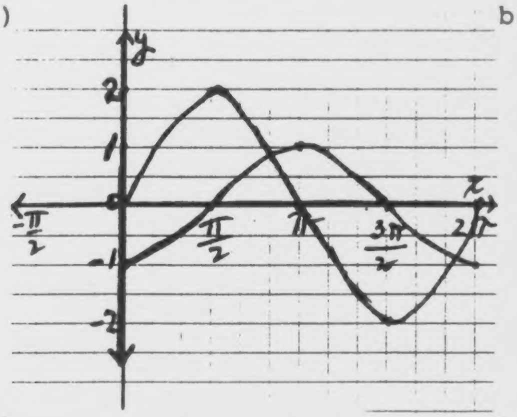


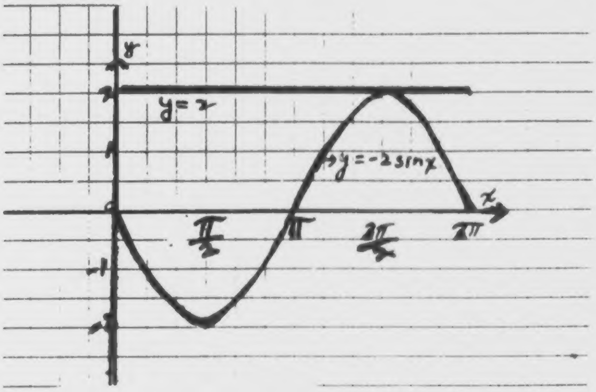
3) b) -10 sin x

4) a)



b) II, IV

5) a)



b) 1) 0

2) 1

SUMMARY:

1. How is the graph of y = 6sin x different from the graph of y = sin x?
2. What points must be indicated on the sketch of a trigonometric function?
3. What is the difference between the graphs of y = 6cos x and y = -6cos x?

LESSON: 3

AIM: How do we sketch the graphs of $y = a \sin bx$ and $y = a \cos bx$?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Find the amplitude and period of $y = a \sin bx$ and $y = a \cos bx$.
2. Sketch the graphs of $y = a \cos bx$ and $y = a \sin bx$ using values of b equal to 1, $\frac{1}{2}$ and 2.
3. State the equation of the trigonometric function when given either a) the amplitude and the period or b) the graph of the equation.
4. Sketch the graphs of $y = a \sin bx$ and $y = a \cos bx$ on the same set of axes and identify the points of intersection.

MOTIVATION:

How is 2 sin x different from sin 2x?

Students should understand that the 2 in 2 sin x multiplies sin x and the 2 in sin 2x multiplies the angle. State that the aim of the lesson is to learn how to sketch trigonometric functions of the form $y = a \sin bx$ and $y = a \cos bx$.

DEVELOPMENT:

1. Have the class fill in the table below.

x	y = sin 2x	y
0		
$\frac{\pi}{2}$		
π		
$\frac{3\pi}{2}$		
2π		

Leave room for entries as shown in the table.

The values for y in the table are all zero. Students should understand that the graph of this function is a sine wave which must reach maximums and minimums. Ask the class to find the value(s) of x for which sin 2x reaches a maximum. The reason for providing a space between 0 and $\frac{\pi}{2}$ should now be evident because sin 2x reaches a maximum when $x = \frac{\pi}{4}$ which is halfway between 0 and $\frac{\pi}{2}$. Have the students fill in the remaining missing values and sketch the graph of y = sin 2x. Have the class reach the conclusion that the amplitude is one and the period is π. The effect of the 2 is to divide the period by 2.

2. Now have the class sketch $y = \sin \frac{1}{2}x$ by filling in the following table:

x	$y = \sin \frac{1}{2}x$	y
0		
$\frac{\pi}{2}$		
π		
$\frac{3\pi}{2}$		
2π		

Pupils should understand that the $\frac{1}{2}$ had the effect of doubling the period.

3. At this point state the following rules:
   For y = a sin bx and y = a cos bx
   amplitude = $|a|$
   period = $\frac{2\pi}{b}$

   You may also state that the frequency is the number of periods completed in 2π radians. The frequency in each case is b.

4. Explain that students will be required to sketch trigonometry functions in which the value of b is limited to $\frac{1}{2}$, 1 and 2 but should be able to state the period for all values of b.

5. In order to sketch a function of the form y = a sin bx and y = a cos bx have your students do the following:
   1) Find the amplitude and period.

2) Indicate values on the y-axis going from +a to -a.

3) On the x-axis, divide the interval form 0 to $\frac{2\pi}{b}$ (period) into 4 equal parts.

4) Have your students graph one period of the function using the intervals indicated on the axes. After students have mastered sketching functions of this type, then assign problems in which the interval on the x-axis does not match the period.

APPLICATIONS:

1. State the amplitude and period of each of the functions.
   a) $y = 4\cos 3x$ b) $y = 2\sin \frac{1}{2}x$ c) $y = \cos 2x$
   d) $y = -\sin \frac{1}{4}x$ e) $y = -4\cos x$

2. a) On the same set of axes sketch the graphs of $y = 2\sin x$ and $y = \cos 2x$ over the interval $0 \le x \le 2\pi$.
   b) For how many values of x in the interval $0 \le x \le 2\pi$ does $2 \sin x = \cos 2x$?

3. a) On the same set of axes sketch $y = 2\cos \frac{1}{2}x$ and $y = -\sin x$ over the interval $0 \le x \le 2\pi$.
   b) For what value of x in the interval $0 \le x \le 2\pi$ does $2\cos \frac{1}{2}x = -\sin x$?

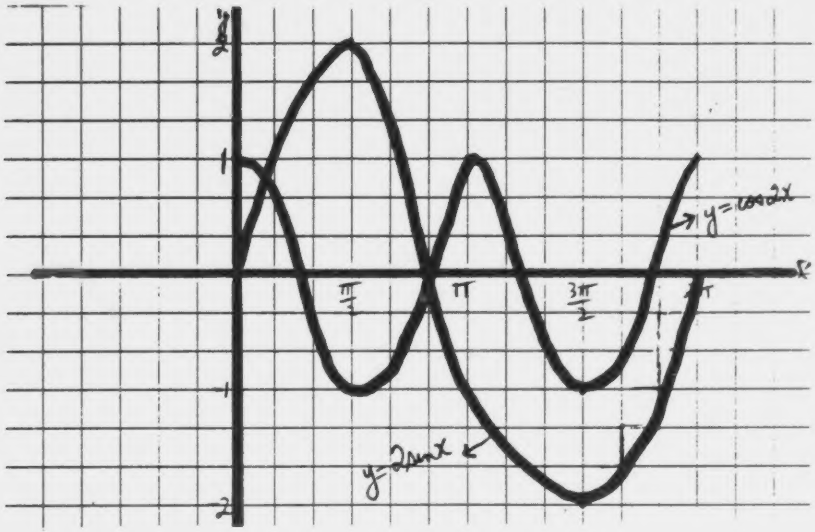
4. a) On the same set of axes sketch $y = \sin x$ and $y = 2\cos 2x$ over the interval $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.
   b) For how many values in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$ does $\sin x = 2\cos 2x$.

SOLUTIONS TO APPLICATION PROBLEMS:

1. a) $a = 4$
   $p = \frac{2\pi}{3}$
   b) $a = 2$
   $a = 4\pi$
   c) $a = 1$
   $p = \pi$
   d) $a = 1$
   $p = 8\pi$
   e) $a = 4$
   $p = 2\pi$

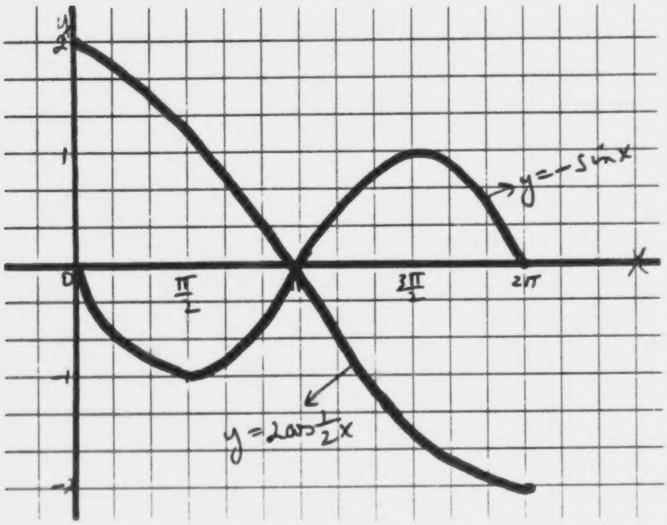
2. a)

b) 1

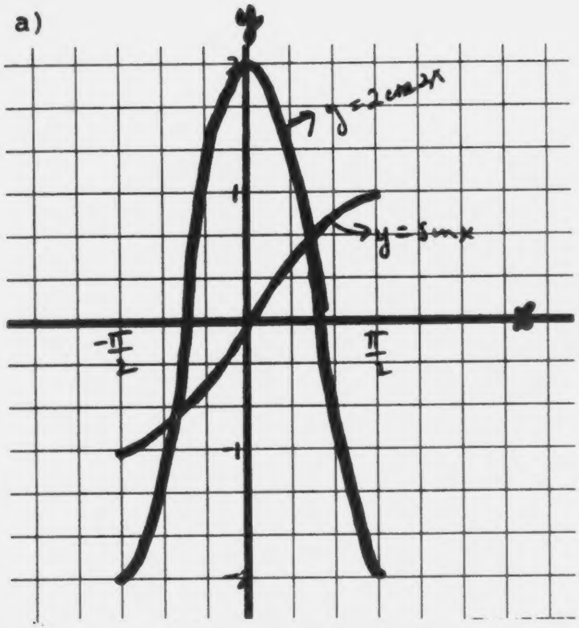

y
2
1
-1
-2
x
π/2
π
3π/2
2π
y = cos 2x
y = 2 sin x

3. a)

b) x = π


y
2
1
-1
-2
0
π/2
3π/2
2π
y = -sin x
y = 2 cos ½x

4. a)



b) 2

SUMMARY:

1. What is the amplitude and period of y = a sin bx and y = a cos bx?
2. What are the steps in sketching y = a sin bx and y = a cos bx?
3. In the equation y = a sin bx explain why "b" changes the period?

LESSON: 4

AIM: How do we sketch the graph of $y = \tan x$?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Sketch the graph of $y = \tan x$.
2 Sketch the intersection of $y = \tan x$ with a) $y = \sin bx$ and b) $y = a \cos bx$.

VOCABULARY: asymptote

MOTIVATION:

Ask the students to answer the following question: Using your trigonometric table, describe what change takes place in tan x as x varies from 0 to $\frac{\pi}{2}$. Students should find that the values for tan x begin at zero, increase to one at $\frac{\pi}{2}$ radians and then become extremely large as x approaches $\frac{\pi}{2}$. You should then explain that the aim of the lesson is to sketch $y = \tan x$.

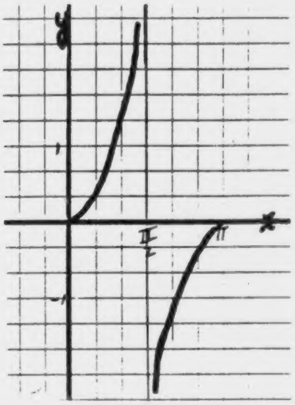
DEVELOPMENT:

1. Begin by asking the class to find the following:
   a) $\tan 0$
   b) $\tan \frac{\pi}{4}$
   c) $\tan 89°$
   d) $\tan 89°10$

From these answers set up the axis for $y = \tan x$ as shown below and sketch $y = \tan x$ from $x = 0$ to $x = \frac{\pi}{2}$. Make sure that students know that the ordered pair $\left(\frac{\pi}{4}, 1\right)$ must be shown in the sketch.

2. In order to sketch $y = \tan x$ from $\frac{\pi}{2} \leq x \leq \pi$ have students find the following values:
   a) $\tan \pi$
   b) $\tan \frac{3\pi}{4}$
   c) $91°$
   d) $\tan 90°50$

From these answers, students should see that tan x increases from - to 0 an x increases from $\frac{\pi}{2}$ to π. The full sketch is drawn below.



Discuss the following with the class:

1) amplitude is undefined and the period is π.
2) There is an asymptote at $x = \frac{\pi}{2}$.

The graph of $y = \tan x$ approaches the asymptote but never meets the asymptote.

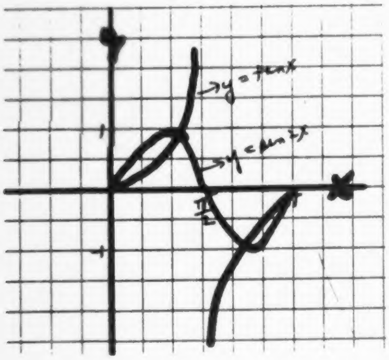
Complete the graph of $y = \tan x$ over the interval $0 \leq x \leq 2\pi$.

APPLICATIONS:

1. a) On the same set of axes sketch the graphs of $y = \tan x$ and $y = \sin 2x$ over the interval $0 \leq x \leq \pi$.
   b) For what values of x in the interval $0 \leq x \leq \pi$ does $\tan x = \sin 2x$.
2. a) On the same set of axes sketch the graphs of $y = \tan x$ and $y = 2\cos x$ from x = to 0 to $x = 2\pi$.
   b) For how many values in the interval $0 \leq x \leq 2\pi$ does $\tan x = 2\cos x$.

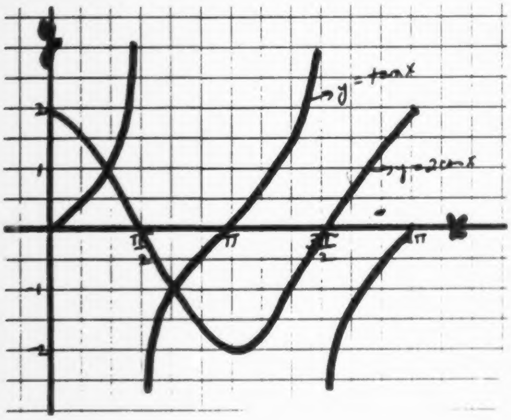
SOLUTIONS TO APPLICATION PROBLEMS:

1. (a)



(b) $x = \frac{\pi}{4}$, $x = \frac{3\pi}{4}$

2. (a)



(b) 2 values

SUMMARY

1. What is the amplitude and period of y = tan x?
2. As x increases towards $\frac{\pi}{2}$, what value does tan x approach? How is this demonstrated in the graph of y = tan x?
3. What is the range of y = tan x?
4. What is an asymptote?

LESSON: 5

AIM: How do we find the cotangent, cosecant and secant of an angle?

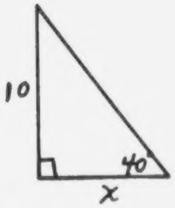
PERFORMANCE OBJECTIVES: The student will be able to...

1. State the definitions of the cotangent, cosecant and secant of an angle.
2. Express the cotangent, cosecant and secant of an angle in terms of their reciprocals.
3. Evaluate the functions of the special angles.
4. Find the other 5 trigonometric functions of an angle given the value of one of the functions.

VOCABULARY: cotangent ratio, consecant ratio, secant ratio

MOTIVATION:

Find the length of x.



DEVELOPMENT:

1. Using the motivation problem, ask "What trigonometric function would we use to find the length of x?" Elicit: The tangent ratio. Ask: "What is the proportion used to find x?" Elicit: Tan 40° = $\frac{10}{x}$.
   Ask: "What difficulty do we have in solving this proportion?" Elicit: After looking up the tan 40° and solving the equation for x, we would have to divide 10 by .8391. Have students calculate the answer (11.9).

2. Ask: "What ratio would we like to see instead of $\frac{10}{x}$ to avoid dividing?" Elicit: $\frac{x}{10}$. Ask: "What is the relationship between the two fractions?" Elicit: They are reciprocals.

Establish that if the tangent ratio is defined as

$$\frac{\text{the length of the side opposite the acute angle}}{\text{the length of the side adjacent to the acute angle}}$$

perhaps there is a trigonometric ratio which represents the reciprocal

$$\frac{\text{the length of the side adjacent to the acute angle}}{\text{the length of the side opposite the acute angle}}$$

Introduce the **cotangent (cot)** ratio as the reciprocal of the tangent ratio

$$\tan A = \frac{1}{\cot A} \qquad \cot A = \frac{1}{\tan A}$$

$$\tan A \cdot \cot A = 1$$

Verify the motivation problem by using the cotangent ratio.

3. Do Application #1.

4. Ask: "What is the definition of the sine ratio?"
   Elicit:

$$\frac{\text{the length of the side opposite the acute angle}}{\text{the length of the hypotenuse}}$$

Ask: "What would the reciprocal of the sine be equal to?

Elicit: $$\frac{\text{the length of the hypotenuse}}{\text{the length of the side opposite the acute angle}}$$

Introduce the **cosecant (csc)** ratio as the reciprocal of the sine ratio.

$$\sin A = \frac{1}{\csc A} \qquad \csc A = \frac{1}{\sin A}$$

$$\cos A \cdot \sec A = 1$$

5. Do Application #2.

6. Ask: "What is the definition of the cosine ratio?"
   Elicit:

$$\frac{\text{the length of the side adjacent to the acute angle}}{\text{the length of the hypotenuse}}$$

Ask: "What would be the reciprocal of the cosine be equal to?"

Elicit: $$\frac{\text{the length of the hypotenuse}}{\text{the length of the side opposite the acute angle}}$$

Introduce the **secant (sec)** ratio as the reciprocal of the cosine ratio.

$$\cos A = \frac{1}{\sec A} \qquad \sec A = \frac{1}{\cos A}$$

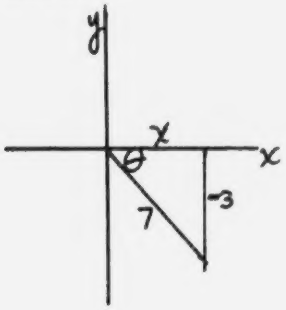
$$\cos A \cdot \sec A = 1$$

7. Do Application #3.

8. Ask: "If $\sin A = 3/5$, what is the csc A equal to?" Elicit: 5/3. Ask: "Why?" Elicit: csc A is the reciprocal of sin A. Ask: "Evaluate sec 60°?" Elicit sec 60° is the reciprocal of cos 60°. If $\cos 60° = 1/2$, then $\sec 60° = 2$.

9. Do Application #4.

10. Ask: "If $\csc \theta = 7/3$ and θ is in quadrant IV, what is the value of the other five trigonometric functions?" Elicit:



$$a^2 + b^2 = c^2$$
$$x^2 + 9 = 49$$
$$x^2 = 40$$
$$x = \sqrt{40}$$

$\sin \theta = -3/7$

$\cos \theta = \sqrt{40}/7$

$\sec \theta = 7/\sqrt{40}$

$\tan \theta = -3/\sqrt{40}$

$\cot \theta = -\sqrt{40}/3$

11. Do Application #5.

APPLICATIONS:

1. Given the value of tan A in each of the following, find cot A:
   a) 2/3 b) 1/4 c) 0 d) x

2. In each of the following find the csc A if sin A equals:
   a) $\sqrt{2}/3$ b) 1 c) $1/a$, $a \neq 0$ d) $2 - \sqrt{2}$

3. In each of the following find the cos A if sec A equals:
   a) $1/\sqrt{3}$ b) .2 c) x^2

4. Find the exact numerical value of each expression.

   a) $\csc 150^\circ$ b) $\sec 225^\circ$ c) $\cot \pi/4$

   d) $\sin 30^\circ \cdot \csc 30^\circ$

5. If $\sin A = 3/5$ and $\cos A < 0$, find each of the following:

   a) $\cos A$ b) $\tan A$ c) $\sec A$ d) $\csc A$ e) $\cot A$

SOLUTIONS TO APPLICATION PROBLEMS:

1. a) $3/2$ b) 4 c) undefined d) $1/x,\ x \neq 0$
2. a) $3/\sqrt{2}$ b) 1 c) a d) $\frac{2+\sqrt{2}}{2}$
3. a) $\sqrt{3}$ b) 5 c) $1/x^2,\ x \neq 0$
4. a) $1/2$ b) $-\sqrt{2}/2$ c) 1 d) 1
5. a) $-4/5$ b) $-3/4$ c) $-5/4$ d) $5/3$ e) $-4/3$

SUMMARY:

1. What is the definition of the cotangent ratio?
2. What is the definition of the cosecant ratio?
3. What is the definition of the secant ratio?

LESSON: 6

AIM: How do we use the Quotient and Pythagorean identities to simplify trigonometric identities?

PERFORMANCE OBJECTIVES: The student will be able to...

1. State the Quotient and Pythagorean identities.
2. Verify the identities using special angles.
3. Express trigonometric functions in terms of sine and cosine.

VOCABULARY: Quotient identity, Pythagorean identity

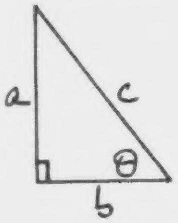
MOTIVATION:

Show that each of the following is true if $\theta = 45°$

a) $\tan\theta = \frac{\sin\theta}{\cos\theta}$ b) $\sin^2\theta + \cos^2\theta = 1$

DEVELOPMENT:

1. Using the motivation problem, students should demonstrate in a)



$$\tan 45° \overset{?}{=} \frac{\sin 45°}{\cos 45°}$$

$$1 \overset{?}{=} \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} \quad \text{true}$$

and b) $(\sin 45°)^2 + (\cos 45°)^2 \overset{?}{=} 1$

$$\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 \overset{?}{=} 1$$

$$2/4 + 2/4 = 1$$

$$1 = 1 \quad \text{true}$$

2. Ask: "Are a) and b) above true for other values of θ?" Elicit 30°, 60°, etc. Ask: "Are they true for all values of θ and if so what kind of equations would they represent?" Elicit: Yes, identities.

3. **Ask:** "How can we prove these identities true?" Elicit: Determine if they hold true in right triangles. (Note: An alternate method of proof would be to use the unit circle and show through line segments that these identities are true.)

Prove: $\operatorname{Tan}\theta \overset{?}{=} \dfrac{\sin\theta}{\cos\theta}$

$$\frac{a}{b} \overset{?}{=} \frac{\frac{a}{c}\cdot c}{\frac{b}{c}\cdot c} = \frac{a}{b}$$

Ask: "If $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ is an identity, what is $\cot\theta$ equal to and why?"

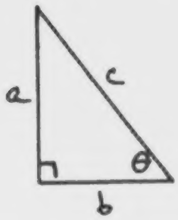
Elicit: $\dfrac{\cos\theta}{\sin\theta}$ because $\cot\theta$ is the reciprocal of $\tan\theta$.

Define **$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$** and **$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$** as

Quotient Identities.

4. Do Application #1.

5. Ask: "Can we prove motivation problem b) an identity in the same way?" Elicit:



$$a^2 + b^2 = c^2$$

Divide by c^2

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{c^2}{c^2}$$

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = \left(\frac{c}{c}\right)^2$$

Ask: "What is the ratio $\frac{a}{c}$ equal to ?" Elicit: $\sin\theta$.

Ask: "What is the ratio $\frac{b}{c}$ equal to ?" Elicit: $\cos\theta$.

Establish $\sin^2\theta + \cos^2\theta = 1$.

6. Ask: "Perhaps there are other identities we can prove using the Pythagorean Theorem, rather than divide $a^2 + b^2 = c^2$ by c^2, what else could we divide by ?" Elicit: a^2 or b^2. Demonstrate $\dfrac{a^2}{a^2} + \dfrac{b^2}{a^2} = \dfrac{c^2}{a^2}$

$\left(\frac{b}{a}\right)^2 = \cot^2\theta$ $\qquad\qquad$ $\frac{c}{a}^{2} = \csc^2\theta$

Therefore $1 + \cot^2\theta = \csc^2\theta$

then $\frac{a^2}{b^2} + \frac{b^2}{b^2} = \frac{c^2}{b^2}$

$\left(\frac{a}{b}\right)^2 = \tan^2\theta$ $\qquad\qquad$ $\left(\frac{c}{b}\right)^2 = \sec^2\theta$

Therefore $\tan^2\theta + 1 = \sec^2\theta$

Define $\boldsymbol{\sin^2\theta + \cos^2\theta = 1}$
$\boldsymbol{1 + \cot^2\theta = \csc^2\theta}$
$\boldsymbol{\tan^2\theta + 1 = \sec^2\theta}$ as Pythagorean Identities.

7. Do Application #2.

8. Do Application #3.

9. Do Application #4.

APPLICATIONS:

1. Write the given expressions as a single function or a constant.
   a) $\sin\theta \cot\theta$ $\quad$ b) $\sin\theta \sec\theta$ $\quad$ c) $\cos\theta \tan\theta \csc\theta$
   d) $\frac{1 + \cos\theta}{\sin\theta} - \frac{\cos\theta}{\sin\theta}$

2. Write the given expression as a single function or a constant.
   a) $\cos\theta(\tan^2\theta + 1)$ $\quad$ b) $\csc\theta \sin\theta - \sin^2\theta$
   c) $\dfrac{\frac{1}{\cos^2} - 1}{\frac{\sin^2\theta}{\cos^2\theta}}$

3. Use a Pythagorean Identity in each of the following:
   a) If $\sin A = 3/5$ and angle A is in quadrant II, find $\cos A$.
   b) If $\sec B = -\sqrt{5}$ and angle B is in quadrant III, find $\tan B$.

4. Express each of the following in terms of $\sin A$ and/or $\cos A$.
   a) $\tan A$ $\quad$ b) $\cot A$ $\quad$ c) $\sec A$ $\quad$ d) $\csc A$

SOLUTIONS TO APPLICATION PROBLEMS:

1. a) $\cos\theta$ b) $\tan\theta$ c) 1 d) $\csc\theta$
2. a) $\sec\theta$ b) $\cos^2\theta$ c) 1
3. a) $-4/5$ b) 2
4. a) $\frac{\sin A}{\cos A}$ b) $\frac{\cos A}{\sin A}$ c) $\frac{1}{\cos A}$ d) $\frac{1}{\sin A}$

SUMMARY:

1. What are the Quotient Identities?
2. What are the Pythagorean Identities?

LESSON: 7

AIM: How do we prove trigonometric identities?

PERFORMANCE OBJECTIVES: The student will be able to...

1. State that a trigonometric equation is an identity when it is true for all values of the given angle.
2. Prove trigonometric identities by expressing all the functions in terms of sine and/or cosine.

VOCABULARY: trigonometric identity

MOTIVATION:

Write the definitions of an equation which is an identity.

DEVELOPMENT:

1. Using the motivation problem review the definitions of an identity as an equation which is true for all values of the variable.
   Give several examples: $3x + 4x = 7x$; $a(b + c) = ab + ac$; $x + 4 = 4 + x$; $\sin^2\theta + \cos^2\theta = 1$.
   Ask: "What is a trigonometric identity?" Elicit: An equation true for all values of the given angle.
2. Introduce $\sin\theta \cot\theta = \cos\theta$. Ask: "How can we prove this equation is an identity?" Elicit: Show that the right side of the equation is exactly the same as the left side of the equation. Ask: "How can we change one or both sides of the equation so that they will be the same?" Elicit: Substitute given functions with previously learned identities.
3. Do Application #1.
4. Demonstrate the similarity between proving an identity and deductive reasoning in a geometric proof.
5. Explain that most trigonometric identities can be solved in several ways but that since we can convert all trigonometric functions to sines and/or cosines, we find it easier to solve most identities in this way.

Elicit the steps needed to prove the identity:

$$\frac{\sec\theta}{\tan\theta} = \csc\theta$$

$$\frac{\cos\theta \cdot \frac{1}{\cos\theta}}{\cos\theta \cdot \frac{\sin\theta}{\cos\theta}} \overset{?}{=} \frac{1}{\sin\theta}$$

$$\frac{1}{\sin\theta} = \frac{1}{\sin\theta}$$

6. Do Application #2.

7. Introduce: Prove the identity $\tan^2\theta(1 + \cot^2\theta) = \sec^2\theta$. Explain that since we are dealing with squares, it might be helpful to use another method. Ask: "What other identities have we learned that we could use to prove this identity?" Elicit: Pythagorean Identities.

8. Do Application #3.

9. Note: Remind students that these equations are to be proved true and therefore we cannot use the properties of equality to prove them true. Only proven substitutions can be used.

APPLICATIONS:

1. Prove the following identities:
   a) $\sec\theta \cot\theta = \csc\theta$
   b) $\cos\theta \tan\theta = \sin\theta$

2. Prove the following identities:
   a) $\frac{\sin\theta}{\cot\theta} + \cos\theta = \sec\theta$
   b) $\sec\theta = \cos\theta + \tan\theta \sin\theta$

3. Prove the following identities:
   a) $\cos^2\theta(\sec^2\theta - 1) = \sin^2\theta$
   b) $\sec^2\theta(1 - \cos^2\theta) = \tan^2\theta$
   c) $(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$
   d) $\tan^2\theta \cdot \csc^2\theta = \sec^2\theta$

SOLUTIONS TO APPLICATION PROBLEMS:

Note: There are often many ways to prove these identities. The teacher should encourage alternate solutions. However, the teacher should also encourage those students who are having difficulty to follow the suggestions set down in this lesson.

SUMMARY:

1. What is a trigonometric identity?
2. What suggestions do we usually follow in proving trigonometric identities?

LESSON: 8

AIM: How do we prove trigonometric identities when it is necessary to simplify complex fractions or combine fractions?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Prove trigonometric identities by simplifying complex fractions.
2. Prove trigonometric identities by combining fractions.

MOTIVATION:

Prove the following identity: $\frac{1 - \csc^2\theta}{1 - \csc\theta} = \frac{1 + \sin\theta}{\sin\theta}$

DEVELOPMENT:

1. Using the motivation problem, elicit from students that they should change csc θ to $\frac{1}{\sin\theta}$ and then have them realize that the left side is now a complex fraction.

$$\frac{1 - \frac{1}{\sin^2\theta}}{1 - \frac{1}{\sin\theta}} = \frac{1 + \sin\theta}{\sin\theta}$$

   Ask: "What kind of fraction is created on the left side?" Elicit: A complex fraction.
   Review the definition of a complex fraction as a fraction which contains a fraction in its numerator, denominator or both. Ask: "How do we simplify a complex fraction?" Elicit: Find the LCD and then multiply both the numerator and denominator by the LCD. Ask: "What is the LCD in this complex fraction?"
   Elicit: sin θ. Complete motivation problem eliciting steps needed.

2. Do Application #1.

3. Introduce $\frac{a}{b} - \frac{b}{a}$ and ask: "How do we combine these two fractions?" Elicit: Find the LCD and multiply the numerator and denominator of each fraction by the same

factor which will produce the LCD in the denominator of each fraction. Ask: "What is the LCD in this problem?" Elicit: ab.

Complete the problem $\frac{a^2 - b^2}{ab}$

Similarly combine $\frac{\sin\theta}{\cos\theta} - \frac{\cos\theta}{\sin\theta}$, LCD = $\sin\theta\cos\theta$

Answer: $\frac{\sin^2\theta - \cos^2\theta}{\sin\theta\cos\theta}$

4. Indicate that sometimes we must use the concept of combining fractions to prove identities.

   Introduce $1 + \sec\theta = \frac{\cos\theta + 1}{\cos\theta}$

   Ask: "What is the first step in proving this identity?"

   Elicit: Change $\sec\theta$ to $\frac{1}{\cos\theta}$

   $$1 + \frac{1}{\cos\theta} = \frac{\cos\theta + 1}{\cos\theta}$$

   Ask: "How would you combine $1 + \frac{1}{\cos\theta}$?" Elicit: Find the LCD = $\cos\theta$ and change 1 into $\frac{\cos\theta}{\cos\theta}$ producing $\frac{\cos\theta}{\cos\theta} + \frac{1}{\cos\theta}$ and combining into $\frac{\cos\theta + 1}{\cos}$ proving the identity.

5. Do Application #2.

APPLICATIONS:

1. Prove the following identities:
   a) $\frac{\sec A}{\csc A} = \tan A$
   b) $(1 + \sec B)/(1 - \sec^2 B) = \cos B/(\cos B - 1)$
   c) $\frac{1 + \csc x}{\sec x} = \cos x + \cot x$

2. Prove the identity:
   $1/(1 + \sin C) + 1/(1 - \sin C) = 2\sec^2 C$

SOLUTIONS TO APPLICATION PROBLEMS:

See note with Solutions to Application Problems in the previous lesson.

SUMMARY:

1. How do we prove trigonometric identities involving complex fractions?
2. How do we prove trigonometric identities involving combining fractions?

LESSON: 9

AIM: How do we solve linear trigonometric equations?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Solve a trigonometric equation for the trigonometric function.
2. Find the reference angle when given the value of the function.
3. Find all the solutions to a trigonometric equation which are greater than or equal to 0° and less than or equal to 360°.

MOTIVATION:

Solve for $\sin \theta$: $2\sin \theta - 1 = 0$

DEVELOPMENT:

1. Using the motivation problem, students will review finding the solution of a linear equation where the variable is $\sin \theta$. Ask: "If $\sin \theta = 1/2$, what acute angle will make the equation true?" Elicit: 30°.
2. Do Application #1.
3. Ask: "Is there any other angle in the interval $0° \leq \theta \leq 360°$ which also has its sine equal to 1/2?" Elicit: Yes, 150°.

   Note: The teacher may have to review finding the reference angle and the possible quadrants that an angle may lie in.
4. Do Application #2.
5. Introduce $\tan \theta = 1/3$. Ask: "Are we aware of any angle whose tangent is equal to 1/3?" Elicit: No.
   Ask: "How can we find the angle, to the nearest degree, whose tangent is 1/3?" Elicit: Look up the decimal .3333 in the trig tables. $\theta = 18°$ to the nearest degree.
6. Do Application #3.

APPLICATIONS:

1. Solve each equation for the positive acute angle.
   - a) $2\cos A - \sqrt{3} = 0$
   - b) $2\cos B - 1 = 0$
   - c) $3\tan C = 3$
   - d) $1 - 8\sin \theta = -3$
2. Solve each equation for all values of x in the interval $0° \leq x \leq 360°$.
   - a) $3\tan x - \sqrt{3} = 0$
   - b) $\cos x = 1$
   - c) $5\tan x + 2 = 3\tan x$
   - d) $2(\sin x + 1) = \sin x + 3$
3. Solve each equation for all values in the interval $0° \leq x \leq 360°$.
   - a) $5\sin x - 1 = 0$
   - b) $7\cos x + 2 = 0$

SOLUTIONS TO APPLICATION PROBLEMS:

1. a) 30°
   b) 60°
   c) 45°
   d) 30°

2. a) 30°, 210°
   b) 0°, 360°
   c) 135°, 315°
   d) 90°

3. a) 12°, 168°
   b) 107°, 253°

SUMMARY:

How do we solve linear trigonometric equations?

LESSON: 10

AIM: How do we solve quadratic trigonometric equations?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Solve quadratic trigonometric equations either by factoring or by using the quadratic formula.
2. Find all the positive angles less than 360° that will satisfy the equation.

MOTIVATION:

Solve the following equation: $x^2 - 3x - 10 = 0$

DEVELOPMENT:

1. Using the motivation problem students will review the solution of quadratic equations by factoring. Ask: "What kind of equation is $x^2 - 3x - 10 = 0$?" Elicit: A quadratic equation. Ask: "What method have we learned to solve this equation?" Elicit: Factoring. Ask: "How do we factor this equation?" Elicit: $(x - 5)(x + 2) = 0$. Ask: "What are the roots of the equation?" Elicit: 5 or -2.
2. Introduce $\sin^2\theta - \sin\theta = 0$. Ask: "What kind of equation is this?" Elicit: A quadratic trigonometric equation. Ask: "Can we factor it and what are the factors?" Elicit: $\sin\theta(\sin\theta - 1)$. Ask: "What are the roots of the equation? Elicit: 0 and 1. Ask: "If $\sin\theta = 0$, and $0 \leq \theta \leq 360°$ what is the value(s) of θ?" Elicit: 0°, 180° and 360°. Ask: "If $\sin\theta = 1$, what is the value(s) of θ?" Elicit: 90°.
3. Do Application #1.
4. Introduce $\tan^2\theta - 5\tan\theta + 4 = 0$. Ask: "What are the values of the interval $0° \leq \theta < 360°$ to the nearest degree?" Elicit: $(\tan\theta - 4)\ (\tan\theta - 1) = 0$

$$\tan\theta = 4 \text{ or } \tan\theta = 1$$
$$\theta = 76° \qquad \theta = 45°$$
$$\theta = 256° \qquad \theta = 225°$$

5. Do Application #2.

6. Introduce $3\cos^2\theta - 5\cos\theta - 4 = 0$. Ask: "Is this equation factorable?" Elicit: No. Ask: "How can we solve the equation?" Elicit: Use the quadratic formula.

   Review the quadratic formula: $\cos\theta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

   Solve the equation. $\cos\theta = \frac{5 \pm \sqrt{73}}{6}$

   $\cos\theta = -.59$

   $\theta = 126° \text{ or } 234°$

7. Do Application #3.

APPLICATIONS:

1. Find all values of θ in the interval $0° \leq \theta < 360°$ that satisfy the given equation.
   a) $\tan^2\theta - 3 = 0$
   b) $2\cos^2\theta - 1 = 0$

2. Find to the nearest degree all values of θ in the interval $0° \leq \theta < 360°$ that satisfy the given equation.
   a) $4\sin^2\theta - 3\sin\theta - 1 = 0$
   b) $2\tan\theta - 7 = \frac{4}{\tan\theta}$

3. Find to the nearest degree all values of θ in the interval $0° \leq \theta < 360°$ that satisfy the given equation.
   a) $4\sin^2\theta - 2\sin\theta - 3 = 0$
   b) $2\tan^2\theta - \tan\theta = 2$

SOLUTIONS TO APPLICATION PROBLEMS:

1. a) 60°, 120°, 240°, 300°
   b) 45°, 135°, 225°, 315°

2. a) 90°, 194°, 346°
   b) 76°, 153°, 256°, 333°

3. a) 221°, 319°
   b) 52°, 142°, 232°, 322°

SUMMARY:

How do we solve quadratic trigonometric equations?

LESSON: 11

Aim: How do we solve trigonmetric equations which contain more than one function?

PERFORMANCE OBJECTIVES: The student will be able to ...

1. Express the equation in terms of one trigonmetric function by substitution.
2. Solve the resulting equation for all values of the angle in the interval 0° to 360° inclusive.

MOTIVATION:

Solve for x in the interval $0° \leq x < 360°$: $\sin x = \frac{1}{\sin x}$

DEVELOPMENT:

1. Using the motivation problem students will get $\sin^2 x = 1$, $\sin x = \pm 1$, $x = 90°$, or $x = 270°$. Ask: "How does the equation $\sin x = \csc x$ compare to the given equation?" Elicit: $\csc x = \frac{1}{\sin x}$. Ask: "How can we solve the equation $\sin x = \csc x$?" Elicit: Substitute $\frac{1}{\sin x}$ for $\csc x$ and solve as in the motivation problem. Review trigonmetric identities.
2. Ask: "Given $2\sin\theta + 2\csc\theta = 5$, what would we have to do before we can solve this equation?" Elicit: Substitute for one of the functions.

$$2\sin\theta + \frac{2}{\sin\theta} = 5$$

$$2\sin^2\theta - 5\sin\theta + 2 = 0$$
$$(2\sin\theta - 1)(\sin\theta - 2) = 0$$
$$\sin\theta = 1/2 \text{ or } \sin\theta = 2$$
$$\theta = 30°, 150° \quad \text{rejected}$$

3. Do Application #1.

4. Introduce $5\sin^2 \theta + 3\cos \theta = 3$. Ask: "What must we do before we can solve this equation?" Elicit: Substitute $1 - \cos^2 \theta$ for $\sin^2\theta$. Elicit that the equation then becomes $5\cos^2 \theta - 3\cos \theta - 2 = 0$.
   Ask: "How can we solve this equation?" Elicit: Use the quadratic formula. Solutions 0°, 114°, 246°

5. Do Application #2.

APPLICATIONS:

1. Find to the nearest degree, all values of θ in the interval $0° \leq \theta < 360°$ that satisfy the given equation.
   a) $2\cos \theta + 1 = \sec \theta$
   b) $7\sin \theta + 1 = 6\csc \theta$

2. Find to the nearest degree, the values of θ in the interval $0° \leq \theta < 360°$ that satisfy the given equation.
   a) $2\sin^2 \theta + 3\cos \theta = 0$
   b) $2\sec^2 \theta - 3\tan \theta - 1 = 0$

SOLUTION TO APPLICATION PROBLEMS:

1. a) 60°, 180°, 300°
   b) 59°, 121°, 270°

2. a) 120°, 240°
   b) 27°, 45°, 207°, 225°

SUMMARY:

How can we solve trigonmetric equations which contain more than one trigonmetric function?

LESSON: 12

AIM: How do we solve trigonometric equations to the nearest minute?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Calculate the function of angles to the nearest minute.
2. Calculate an angle to the nearest minute given the value of the function.
3. Solve trigonometric equations for all values of the angle in the domain from 0° to 360° to the nearest minute.

VOCABULARY: interpolation

MOTIVATION:

Find the value of sin 34°20', sin 34°30', sin 34°25'.

DEVELOPMENT:

1. Using the motivation problem students will determine that sin 34°25' lies half way between 34°20' and 34°30' whose values we can look up in the tables. Ask: "How will we determine the sin 34°25'?" Elicit: Find the midpoint between .5640 and .5664 which is .5652.
2. Develop the proportion $\frac{5}{10} = \frac{x}{24}$ (difference between values.)

This procedure is called **interpolation** and is used within small intervals.

Demonstrate:

		0	sin θ		
10		34°30'	.5664		24
	5	34°25'	.5640+x	x	
		34°20'	.5640		

$$\frac{5}{10} = \frac{x}{24}\ ;\ x = 12$$

Note: The cosine and cotangent functions have an inverse relationship to their angles and must be considered in evaluating the proportion. See Development part 4.

3. Do Application #1.

4. Ask: "If $\cos\theta = .9583$, find the measure of acute angle θ to the nearest minute?" Elicit:

$$\begin{array}{cc|cc l}
 & 0 & \cos\theta & & \\ \hline
10 & 16^\circ 40' & .9580 & 8 & \leftarrow \text{from the table} \\
x & 16^\circ 30'+x & .9583 & 5 & \\
 & 16^\circ 30' & .9588 & & \leftarrow \text{from the table}
\end{array}$$

$$\frac{x}{10} = \frac{5}{8}\ ;\ x \approx 6$$

Therefore $\theta = 16^\circ 30' + 6' = 16^\circ 36'$.

5. Do Application #2.

6. Introduce $4\cos\theta = 3$. Ask: "What is the positive acute angle to the nearest minute that satisfies the equation?" Elicit: Solve: $\cos\theta = .7500$. Using interpolation.

$$\begin{array}{cc|cc}
 & 0 & \cos\theta & \\ \hline
10 & 41^\circ 30' & .7490 & 19 \\
x & 41^\circ 20'+x & .7500 & 9 \\
 & 41^\circ 20' & .7509 &
\end{array}$$

$$\frac{x}{10} = \frac{9}{19}\ ;\ x \approx 5\ ;\ \theta = 41^\circ 25'$$

7. Do Application #3.

APPLICATIONS:

1. Using interpolation, find the value of the trigonometric function to four decimal places.
   a) tan 38°13'
   b) cos 53°46'

2. Use interpolation to find the measure of acute angle A to the nearest minute.
   a) tan A = .1175
   b) cos A = .5840

3. Find the measure of acute angle A to the nearest minute.
   a) $5\cos A - 1 = 0$
   b) $1 - 3\sin A = 0$

SOLUTIONS TO APPLICATION PROBLEMS:

1. a) .7874
   b) .5911
2. a) 6°42'
   b) 54°16'
3. a) 78°28'
   b) 19°28'

SUMMARY:

How do we solve trigonometric equations to the nearest minute?

LESSON: 13

AIM: How do we evaluate expressions involving rational exponents:

PERFORMANCE OBJECTIVES: The student will be able to...

Evaluate expressions of the form:

1. a^n where n is an integer
2. $x^{\frac{1}{n}}$ where n is a non-zero integer, $x > 0$
3. $x^{\frac{m}{n}}$ where m and n are non-zero integers, $x > 0$

VOCABULARY: exponent, rational exponent

MOTIVATION:

For each expression, a) evaluate b) explain the "rule" you used.

1. 2^5
2. 4^0
3. 3^{-2}

Elicit 1. $2^5 = 2\cdot2\cdot2\cdot2\cdot2 = 32$ The **definition** of 2^n, where n is a natural number, tells us to write n factors of a certain number, all factors being 2.

2. $4^0 = 1$. We **define** a^0 to be 1 (as long as $a \neq 0$) because such expressions as $\frac{4^5}{4^5}$ turn out to be 4^0 or 1 depending on how you evaluate it. Thus for consistency, we defined a^0 this way in Course I.

3. $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$. Since $\frac{3^{10}}{3^{12}}$ gives 3^{-2} or $\frac{1}{3^2}$ depending on how you evaluate it, we defined a^{-n} to be $\frac{1}{a^n}$ ($a \neq 0$) in Course I.

KEY QUESTION: Ask, "What other questions can we ask concerning types of exponents?"

DEVELOPMENT:

1. Elicit that we may want to evaluate such expressions as $(3)^{1/2}$, $5^{.8}$, $(125)^{-2/3}$, $5^{\sqrt{2}}$, 2^{π}. The last two examples will be considered in calculus courses; the first few will be considered today.

   Write on the board: **What do we mean by $(3)^{1/2}$?**

   Ask, "If our previous work on the Law of Exponents is to be consistent, what must we get when we multiply $(3)^{1/2} \cdot (3)^{1/2}$?"

   Elicit: 3^1 or 3. Thus $(3)^{1/2} \cdot (3)^{1/2} = 3$

   Ask, "In the past, what number multiplied by itself gave 3?"

   Elicit: 3.

   Write on board:

   Definition 1: If $x \geq 0$, then $(x)^{1/2} = \sqrt{x}$

2. a) Ask, "Why do we insist that $x \geq 0$ in this definition?"

   b) Do Application 1a, b, and c.

   c) Ask, "Why don't we accept -3 as $(9)^{1/2}$ since $(-3)(-3)=9$?"

   Elicit (or inform the students) that we want $(9)^{1/2}$ to be between 9^0 and 9^1. In other words, $9^0 < (9)^{1/2} < 9^1$. Thus we reject the negative possibility since: $1 < (9)^{1/2} < 9$

3. a) Ask, "How do you think we should define $(x)^{1/3}$?"

   Elicit: $(x)^{1/3} = \sqrt[3]{x}$ since $(x)^{1/3} \cdot (x)^{1/3} \cdot (x)^{1/3} = x^1$. This leads to:

   DEFINITION 2: If $x \geq 0$ and n is a natural number, then $x^{1/n} = \sqrt[n]{x}$

   b) Do Application 2a-e. Go over 2d and 2e carefully.

4. a) Ask "What can we mean by $(8)^{2/3}$?"

   METHOD ONE: $(8)^{2/3} \,.\, (8)^{2/3} \,.\, (8)^{2/3} = 8^2 = 64$

   Thus $(8)^{2/3} = \sqrt[3]{8^2}$

   METHOD TWO: $(8)^{2/3} = (8)^{1/3} \,.\, (8)^{1/3}$ (for the Law of Exponents to hold) $(8)^{2/3} = ((8)^{1/3})^2 = (\sqrt[3]{8})^2 = 2^2 = 4$

   Note that BOTH methods yield the same result.

b) Have class find $(16)^{3/4}$

c) Thus we define:

DEFINITIONS 3:

If $x \geq 0$ and m and n are natural numbers, then $(x)^{\frac{m}{n}} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$

Note - the last expression (which results from Method Two) is usually the EASIER ONE to deal with since taking roots first avoids very large numbers. Elicit: Numerator of the rational exponent indicates the power of the base and the denominator of the rational exponent indicates the root to be taken.

5. Have class do applications 3 and 4* and 5*.

APPLICATIONS:

1. Evaluate
   a) $(25)^{1/2}$
   b) $(16)^{-1/2}$
   c) $(9)^{1/2}$

2. Evaluate
   a) $(64)^{1/6}$
   b) $(16)^{1/4}$
   c) $(27)^{1/3}$
   d) $(-8)^{1/3}$
   e) $(125)^{-1/3}$

3. Evaluate
   a) $(8)^{4/3}$
   b) $(81)^{3/4}$
   c) $(1000)^{-1/3}$
   d) $(36)^{1.5}$
   e) $(5^0+8)^{-3/2}$
   f) $(-216)^{2/3}$
   g) $(5)^{3/2}$
   h) $(729)^{2/3}$

*4. Rewrite using fractional exponents:
a) $\sqrt{100^3}$
b) $\sqrt[3]{8^5}$
c) $(\sqrt{27})^4$
d) $\sqrt[4]{9^5}$

*5. Is 3^x always greater than 2^x? Explain

SOLUTIONS TO APPLICATION PROBLEMS:

1. a) 5
b) $\frac{1}{4}$
c) 3

2. a) 2
b) 2
c) 3
d) -2
e) $\frac{1}{5}$

Note: If n is odd $(x)^{1/n}$ IS defined (see ex 2d). Note also that in (e) $(125)^{-1/3} = +\frac{1}{5}$. The negative tells us to place $(125)^{1/3}$ in the denominator. BEWARE - the negative does NOT become part of the answer.

3. a) 16
b) 27
c) $\frac{1}{10}$
d) 216
e) $\frac{1}{27}$
f) 36
g) $5\sqrt{5}$ or $\sqrt{125}$
h) 81

4. a) $(100)^{3/2}$
b) $(8)^{5/3}$
c) $(27)^{4/2}$ or $(27)^{2/1}$ or 27^2
d) $(9)^{5/4}$

5. No: $3^x > 2^x$ if $x > 0$
$3^x = 2^x$ if $x = 0$
$3^x < 2^x$ if $x < 0$ example: $(3)^{-2} < (2)^{-2}$
Since $\frac{1}{9} < \frac{1}{4}$

SUMMARY:

Ask: 1. "What do we mean by $(1000)^{1/20}$?"

2. "What do we mean by $(256)^{-5/2}$?"

Note: Ex. 2 summarizes EVERYTHING:
The negative exponent tells us to place everything in the denominator. The denominator of the rational exponent tells us to take the square root of 256 and the numerator of the rational exponent tells us to raise the result to the 5th power.

$$(256)^{\frac{-5}{2}} = \frac{1}{(\sqrt{256})^5} = \frac{1}{16^5}$$

LESSON: 14

AIM: How do we write numbers using scientific notation?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Write a decimal number in scientific notation.
2. Change a number from scientific notation to ordinary decimal form.
3. Use scientific notation to do a long calculation problem with multiplication and division.
4. To find the number of significant digits in a given number.

VOCABULARY: scientific notation

MOTIVATION:

At a certain time, the sun is approximately 92,800,000 miles from Earth. Write 92,800,000 as a product of 9.28 and a power of 10. (Elicit: 9.28×10^7)

DEVELOPMENT:

1. In science, many numbers (such as the above) have MANY zeros. Let us look at two other scientific examples:

EXAMPLE 1: The velocity of light is 2.99 X 10 cm/sec. Express this as an integer. (29,900,000,000).

EXAMPLE 2: The mass of an electron is 9.12×10^{-28} grams. Rewrite this number in the usual "decimal" form. (0.000 000 000 000 000 000 000 000 000 912)

DEFINITION: A number is said to be in Scientific Notation if it is written in the form $a \times 10^n$ where $1 \leq a \leq 10$ and n is an integer.

2. Use the three above examples to develop

PROCEDURE FOR CHANGING A DECIMAL NUMBER INTO SCIENTIFIC NOTATION.

a) Place the decimal point to the right of the first non-zero digit (this gives "a" in $a \times 10^n$).

b) To get the power of 10, ask "How many places must the decimal move in the number we just wrote to get the original number?" Place a NEGATIVE in the exponent if you had to count to the left. Place a ZERO in the exponent if NO movement is necessary. Place a positive number in the exponent if the movement was to the right.

3. Model Examples:

Ex. 1) Change 165,800,000 to scientific notation

a) a = 1.658

b) 1.65800000. (8 places must be moved to the right to get the original number, so n = + 8.) Thus 165,800,000 = 1.658 X 10^8

Ex. 2) Change 0.0001234 to scientific notation

a) a = 1.234

b) .0001.234 (The decimal must move 4 places to the left to get the original number, so n = 4.) Thus .0001234 = 1.234 x 10^{-4}

4. Do applications 1-9 followed by applications 10, 11

5. **ADVANTAGES OF SCIENTIFIC NOTATION**

1. **CONVENIENCE - We can dispense with many zeros.** (Which is easier to remember 9 x 10^{-28} or the number written with all its zeros?)
2. **EASY TO COMPARE NUMBERS.** (See examples 8 and 9 in applications.)
3. **EASY TO MULTIPLY AND DIVIDE.** (See examples 10 and 11 in applications.)

6. A city estimates its population to be 620,000. Write this number in scientific notation. When we write 6.2 x 10^5 we mean the number is between 6.15 and 6.25 x 10^5 or 615,000 and 625,000 (The zeros are considered to be approximations). Thus we say that the 6 and 2 are the only significant digits in the number. **The number of significant digits is the number of digits in the "a" portion of the number written in scientific notation.**

Note: a) 2.99×10^{10} has 3 significant digits. The number is between 2.985 and 2.995×10^{10}

b) 3.0×10^{10} has 2 significant digits. The number is between 2.95 and 3.05×10^{10}.

c) 3×10^{10} has 1 significant digit. The number is between 2.5 and 3.5×10^{10}. This number is less precise than the number in b) The more significant digits - the more precise the number is and the less error we have in using the number.

7. Many calculators and computers use scientific notation to represent numbers too large or small for the screen. For example, one calculator, when asked to calculate 15! displayed $\boxed{1.30767 \quad 12}$ to represent 1.30767×10^{12}. Other calculators may show 130767 E12, the "E" telling us the exponent of 10.

8. Do applications 12-15

APPLICATIONS:

Change to scientific notation

1) 165,940,000

2) 1,000,000,000

3) 6.4120

4) .00003579

Write in decimal notation:

5) 2.35×10^5

6) 1.623×10^{-3}

7) 5.2×10^0

8) Which is larger .0005 or 66×10^{-4}?

9) Which is larger -6.6×10^5 or 6.6×10-5?

10) Approximate $(1.3 \times 10^{-4}) \times (2.1 \times 10^{75})$

11) Approximate $(6.63 \times 10^{-27}) \div (2.74 \times 10^{-19})$

12) Approximate $(1.2 \times 10^{12}) \times (3.5 \times 10^{15})$

13) Approximate $(5.4 \times 10^{-6}) / (4.5 \times 10^{-14})$

14) A "googol" is the number 1 followed by 100 zeros. Rewrite this number in scientific notation.

15) Find the number of significant digits:
a) 6.201×10^{21}
b) 312000
c) 6.0
d) .035
e) -0.0004167

SOLUTIONS TO APPLICATION PROBLEMS:

1) 1.6594×10^{8}

2) 1.0×10^{9}

3) 6.4120×10^{0}

4) 3.579×10^{-5}

5) 235000

6) 0.001623

7) 5.2

8) 5.0×10^{-4}

9 A positive number is always larger than a negative. Thus 6.6×10^{-5} is larger.

10) $(1.3 \times 10^{-4}) \times (2.1 \times 10^{75}) \approx 2.7 \times 10^{71}$

11) $(6.63 \times 10^{-27}) \div (2.74 \times 10^{-19}) \approx 2.42 \times 10^{8}$

12) 4.2×10^{27}

13) 1.2×10^{8}

14) 1.0×10^{100}

15) a) 4
b) 3
c) 2
d) 2
e) 4

SUMMARY:

Ask, "What is meant by writing a number in scientific notation?"

Ask, "What are the advantages of using scientific notation?"

LESSON: 15

AIM: How do we solve equations involving fractions, negative numbers or variables in the exponent?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Solve equations with fractional or negative exponents
2. Solve exponential equations

VOCABULARY: Exponential Equation

MOTIVATION:

Solve for x:

1. $\sqrt{x} = 5$
2. $x^{1/3} = 2$
3. $x^{3/2} = 27$
4. $8 - 2x^{1/3} = 2$
5. $2x^{1/2} + 2 = -4$
6. $5^x = 125$

DEVELOPMENT:

1. Ask, "How are these equations different from most that you dealt with in the past?" Elicit that in examples 1 - 5 the variable is raised to fractional powers. In example 6, the variable appears in the exponent.
2. Elicit that in solving the first five examples - our aim is to get x to some power alone on one side.

ORIGINAL EQUATION	EQUIVALENT EQUATION	SOLUTION (Elicit)
1) $\sqrt{x} = 5$	$x^{1/2} = 5$	Squaring gives $x = 25$
2) $x^{1/3} = 2$	$x^{1/3} = 2$	Cubing gives $x = 8$
3) $x^{3/2} = 27$	$x^{3/2} = 27$	Raise both sides to the $\frac{2}{3}$ power: $(x^{3/2})^{2/3} = (27)^{2/3}$; $x = 9$ **Elicit that the power is chosen by taking the RECIPROCAL of the power of x**
4) $8 - 2x^{-1/3} = 2$	$x^{-1/3} = 3$	Raise both sides to the -3 power: $(x^{-1/3})^{-3} = 3^{-3}$ $x = 3^{-3} = \frac{1}{27}$
5) $2x^{1/2} + 2 = -4$	$x^{1/2} = -3$	Squaring gives $x = 9$

3. **Note:** if we check the last example, we get 2

   $2 \cdot (9)^{1/2} + 2 \overset{?}{=} 4$

   $6 + 2 \neq 4$

   Ask, "Why doesn't this answer check?" Recall from your work in logic that in p → q, if p is false, q can be true, yet the entire conditional is true.

   As an example: p: -1=1 Squaring both sides gives q: 1=1 a true statement! These two equations are not equivalent. Write on board: **NOTE: When raising both sides of an equation to the same power, the resulting equation MAY NOT turn out to be equivalent to the original. Thus, any root of the resulting equation MUST be checked in the original since it may not be a root.**

4. Have class check equations 1-3; do Applications 1-4.

5. Ask, "How did you solve the last equation (number 6)?" Elicit that if $5^x = 5^3$ so $x=3$.

   Definition: An equation in which the variable appears in the exponent of a constant is called an **EXPONENTIAL EQUATION.**

   To solve exponential equations we use the principle that **if $a^x=a^y$, then $x=y$. (With exceptions $a=0, -1, 1$.)**
   If possible get the exponential equation in this form.

6. Do Applications 5-10, 11**.

APPLICATIONS:

Solve for x:

1) $9^x = 27$

2) $5^{2x-6} = 625$

3) $3^x = \frac{1}{81}$

4) $32^{x-4} = 8^x$

5) $x^{2/3} = 25$

6) $x^{-3/4} = 8$

7) $(\frac{1}{2})^{3-x} = 8^x$

8) How do you solve $x^{m/n} = k$?

9) How do you solve $a^x = b$?

10) Solve for x: $25x^2 = 5^{1-x}$

** 11) Solve for x: $3^{x+2} = 16^{x-1}$
What difficulty arises here?

SOLUTIONS TO APPLICATION PROBLEMS:

1) $3^{2x} = 3^3$, $x = 1.5$

2) $5^{2x-6} = 5^4$, $x = 5$

3) $3^x = 3^{-4}$, $x = -4$

4) $2^{5x-20} = 2^{3x}$, $x = 10$

5) $x = 125$

6) $x = \frac{1}{16}$

7) $(2)^{x-3} = 2^{3x}$ so $x - 3 = 3x$ and $x = 1.5$

8) a) Raise both sides to the $\frac{n}{m}$ power
   b) Check your answer

9) Rewrite both sides of the equation with like bases. Then set the exponents equal to one another.

10) $(5^2)^{x^2} = 5^{1-x}$ so $2x^2+x-1 = 0$
$(2x-1)(x+1) = 0$ and $x = \frac{1}{2}, -1$

** 11) $3^{x+2} = 2^{4x-4}$ We cannot write both sides with the same base. We can re-examine this problem at the end of this unit.

SUMMARY:

Ask, "What is the difference between the way we solve $(\frac{1}{2})^x = 4$ and $x^{1/2} = 4$?"

Elicit - In the first we aim to get both sides written with the SAME BASE ($2^{-x} = 2^2$ so $x = -2$).

In the second, we raise both sides to the second power (so $x = 16$).

LESSON: 16

AIM: What is an exponential function and how does it compare with other functions?

PERFORMANCE OBJECTIVES: The student will be able to . . .

1. Sketch the graph of $y = a^x$ where $a > o$, $a \neq 1$.
2. Compare exponential graphs with linear or quadratic graphs.

**3. Apply the new work to a population study.

VOCABULARY: Exponential function, exponential curve

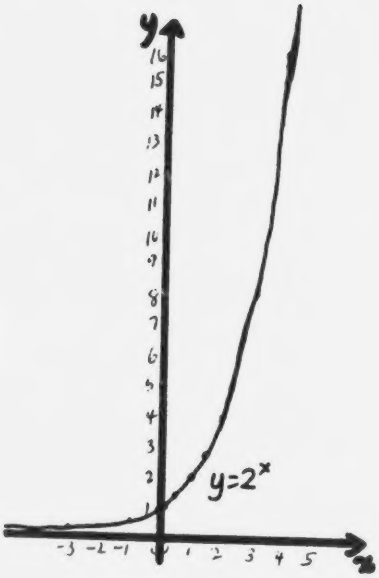
MOTIVATION:

Fill in the following, chart and use it to graph $y = 2^x$.

x	$y = 2^x$
-3	$(\frac{1}{8})$
-2	$(\frac{1}{4})$
-1	$(\frac{1}{2})$
0	(1)
1	(2)
2	(4)
3	(8)
4	(16)
$(\frac{1}{2})$	$(2)^{1/2} = 2 \approx 1.4)$
$(\frac{3}{2})$	$(2)^{3/2} = 8 \approx 2.8)$

DEVELOPMENT:

1.

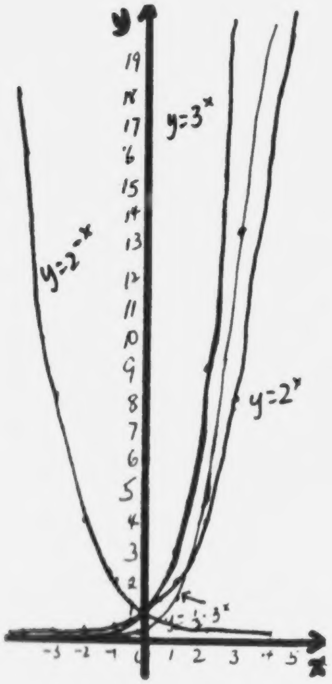


The class should come up with the above graph. Tell them that they have just graphed an "**EXPONENTIAL CURVE.**" Note - the variable x appears in the EXPONENT. Hence this name.

2. **Definition: The** function $y = a^x$ where $a \neq 1$ and $a > o$ is called an **EXPONENTIAL FUNCTION.**

3. Using the same set of axes we just used, let's graph $y = 3x$, $y = (\frac{1}{2})^x$ and $y = \frac{1}{2}(3)^x$. Note $y = (\frac{1}{2})^x$ is the same as $y = 2^{-x}$.
You may wish to assign different rows different equations. Send a representative for each assignment to the board in order to place his (or her) work, using a different color for each curve.

Elicit the following graphs:



4. Elicit the following:

OBSERVATIONS ABOUT EXPONENTIAL CURVES.

1. Exponential curves have equation $y = a^x$, $a \neq 1$, $a > o$
2. The domain is the set of all REAL numbers
3. The range is the set of POSITIVE REAL numbers
4. All exponential curves have a y intercept of 1. (Note: $y = (\frac{1}{2})(3^{x})$ is NOT an exponential equation. We include it here for comparision sake).
5. When a is between 0 and 1 (a fraction) the curve DECREASES (We call that "DECAY"). When $a > 1$, the curve INCREASES (we call that "GROWTH)."
6. These Curves are "asymptotic" to the x-axis. They get "closer and closer" to the x-axis. If $a > 1$, asymptotic to the Negative x-axis. If $0 < a < 1$, asympotitic to the Positive x-axis.

5. Explain to the class that Exponential Curves pop up a great deal in mathematics, biology, demography (populations study), economics etc.

6. Do applications 1 and 2 - Add the following as observation number 7: In exponential growth, the graph looks like a linear graph at first, then increases at a much faster pace than any line.

**7. Do applications 3, 4, and 5.

APPLICATIONS:

1. On the same set of axes sketch $y = 2^x$ and $y = x$

2. On the same of axes, sketch $y = 2^x$ and $y = x^2$. Where do these graphs intersect?

**3. Use this copy of the national U.S. census since 1790 to sketch a graph of U.S. Population Growth:

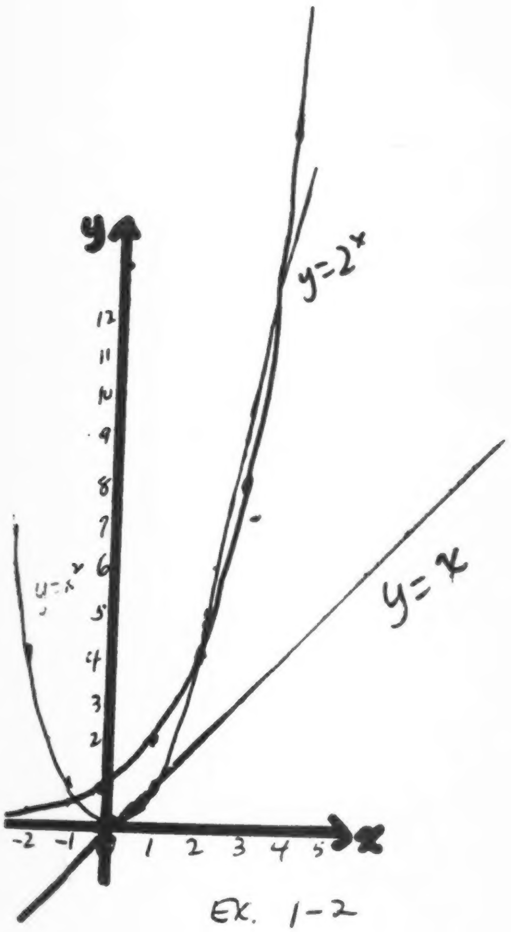
Year	Population	Year	Population
1790	3,929,214	1890	62,947,714
1800	5,308,483	1900	75,994,575
1810	7,239,881	1910	91,972,266
1820	9,638,453	1920	105,710,620
1830	12,866,020	1930	122,775,046
1840	17,069,455	1940	131,669,275
1850	23,191,876	1950	150,697,361
1860	31,443,321	1960	179,323,175
1870	39,818,449	1970	203,302,031
1880	50,155,783	1980	226,549,448

**4. Why should population growth be exponential?

**5. Why is there a slight "dip" in the graph between 1930 and 1940?

SOLUTIONS TO APPLICATION PROBLEMS:

1. See graph below



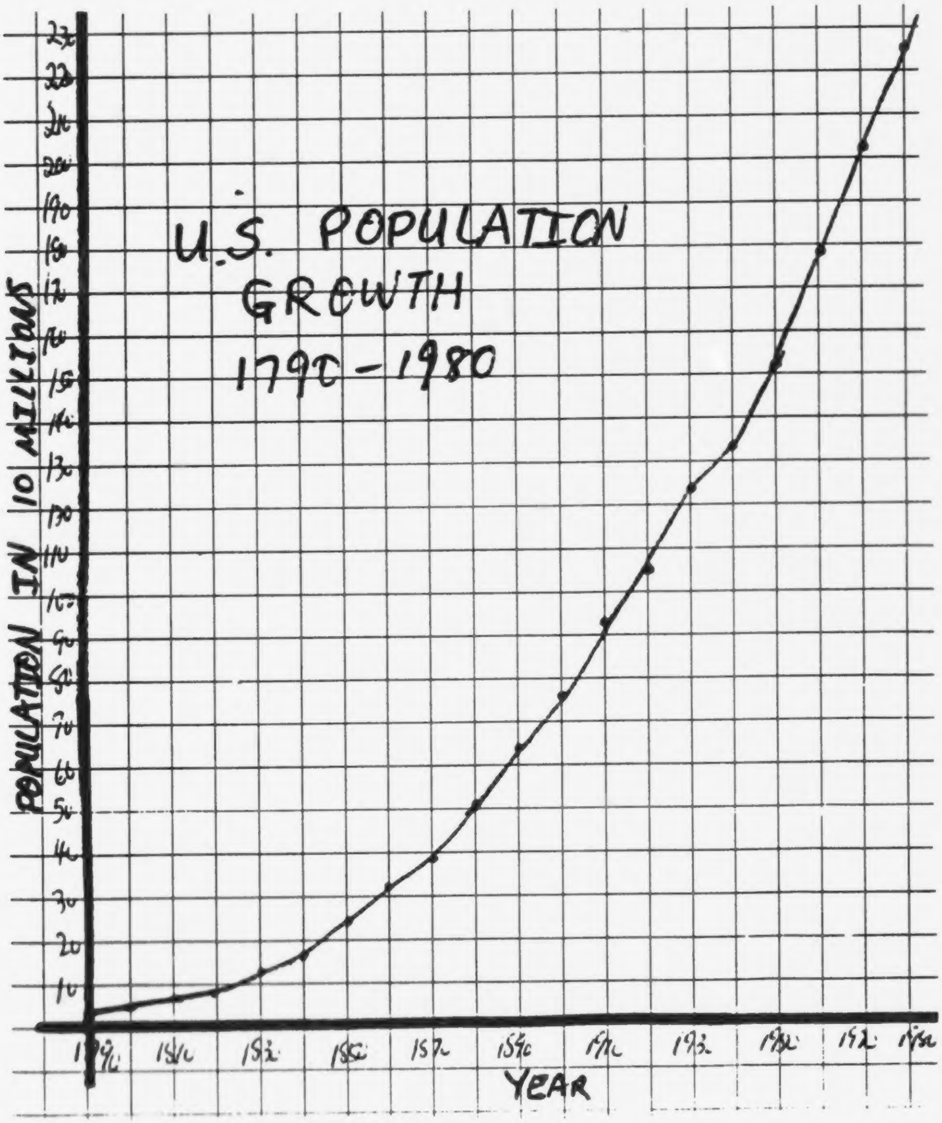
Ex. 1-2

2. Intersection points at (2, 4), (4, 2) and at a third point at which -1 < x < 0 and 0 < y < 1.

**3. See graph below

**4. According to United Nations records, the population of the world hit 4 billion in 1975. The growth rate is approximately a constant 2% yearly. See Below.

EX. 3



4. (continued)

Year	Population of World
1975	4 billion
1976	$4 + 4\ (.02) = 4\ (1.02) = 4.08$
1977	$4(1.02) + .02\ [4(1.02)] = 4(1.02)^2$
1978	$4(1.02)^2 + .02\ [4(1.02)^2] = 4(1.02)^3$
1979	$4(1.02)^3 + .02\ [4(1.02)^3] = 4(1.02)^4$
1980	$4(1.02)^5$
1981	$4(1.02)^6$
.	
.	
.	
1987	$4(1.02)^{12} \approx 5.07$ billion

If y = population and x = the number of years after 1975, then $y = 4(1.02)^x$ (similar to graphs we did earlier).

5. The depression:

SUMMARY:

Ask, "How would you describe what an exponential curve looks like?"

LESSON: 17

AIM: What is the inverse of the exponential function?

PERFORMANCE OBJECTIVES:

1. Sketch the graph of $y = \log_a x$ using the graph of $y = a^x$ $(a > 0, a \neq 1)$.
2. Use the sketches to give the domain and range of both functions.
3. Use properties of inverses and exponents to find corresponding properties for logarithms.

VOCABULARY: Logarithm

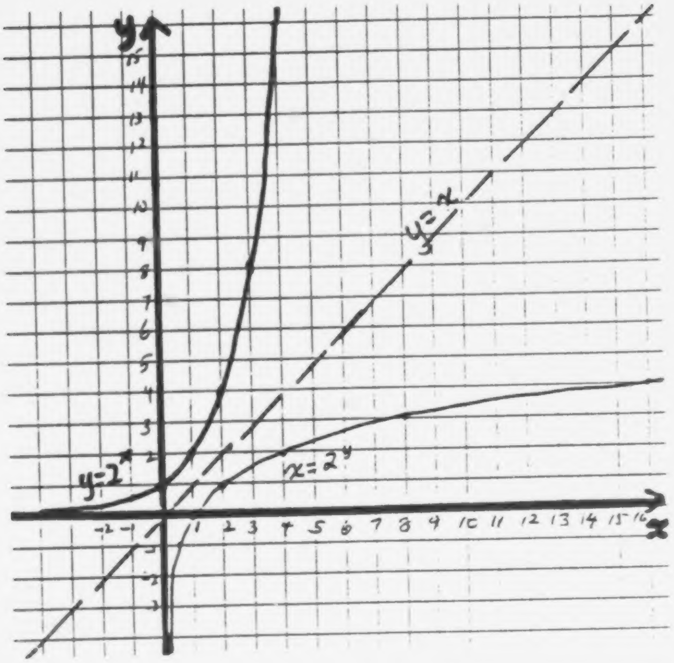
MOTIVATION:

1. On the same set of axes sketch the graphs of
   a) $y = 2^x$ and
   b) the inverse of $y = 2^x$
2. Which of the above two relations are functions?

DEVELOPMENTS:

1. Elicit the two graphs on the next page. Note: Both of the relations $y = 2^x$ and $x = 2^y$ are functions since their graphs pass the vertical line test.
2. Ask, "What geometric transformation is illustrated by these two graphs?" Elicit that $x = 2^y$ is obtained by a reflection of $y = 2^x$ about the line $y = x$. (Recall from earlier work that this ALWAYS happens when you graph a relation and its inverse.
3. Ask, "Is there anything unusual about representing a function with the equation $x = 2^y$?" Elicit that most functions have y solved out. Examples: $y = 2x$, $y = x^2$, $y = \tan 3x$, $y = \sin 2x$ etc. In order to solve out for y in $x = 2^y$, let us focus on "y." Ask, "Can you think of one word to describe y in the equation $x = 2^y$?" Elicit that y is an **EXPONENT**. Up until now, the only times you solved for a variable in an exponent were when BOTH sides

of an equation had the same base. For example, if $3^{2x} = 3^{16}$, then $2x = 16$ and $x = 8$.



In order to solve for y in the equation $x = 2^y$ let us write first: "y is the EXPONENT for the BASE 2 in order to yield x". Now write on the board: **A synonym for "EXPONENT for the BASE 2" is "LOGARITHM to the BASE 2".** Now we can rewrite the original sentence as: $x = 2^y$" is the same as **"y is the logarithm to the base 2 of x."** or more shortly: $y = \log_2 x$. (Write this equation next to its graph.)

4. Elicit some properties using our graphs and write them in the following chart:

PROPERTY OF $y = 2^x$.	CORRESPONDING PROPERTY OF $y = \log_2 x$.
1. The domain is {Real Numbers}. The range is {Positive Reals}.	**1. The domain is {Positive Real Numbers}. The range is {Real Numbers}**
2. The negative x-axis is an asymptote.	**2. The negative y-axis is an asymptote.**
3. The y - intercept is 1.	**3. The x-intercept is 1.**

5. Ask, What is the inverse of $y = 5^x$?" Elicit: $y = \log_5 x$. Ask, "Which of the above properties will be different for $y = 5^x$ and $y = \log_5 x$?" Elicit the above properties are true for ALL exponential and logarithmic functions.

6. Ask, "How would you define $y = \log_b x$? Elicit: **Definition: If $b>0$ and $b \neq 1$ then $y = \log_b x$ means $b^y = x$.** (Remember $y = \log_b x$ means y is the EXPONENT of b to get x.)

7. Do applications 1 and 2.

8. Do applications 3 and 4 if time permits. If the class is above average, do applications *5 and *6.

APPLICATIONS:

1. Explain how to get the graph of $y = \log_5 x$ using another graph.

2. a) What point do all exponential curves have in common.
   b) What point do all logarithmic curves have in common?

3-4 Refer to:
$y = 3^x$
$x = 3^y$
$y = \log_3 x$

3. Which of the above functions are the same?

4. Which of the above functions are inverses of each other?

*5a) We never defined a^x where a is negative and $x = \frac{p}{q}$. Explain why.

*5b) Explain why we cannot take the logarithm of a negative number.

*6) Find $2^{\log_2 9}$

SOLUTIONS TO APPLICATION PROBLEMS:

1. $y = \log_5 x$ is the INVERSE of $y = 5^x$. We can either
   (a) Sketch $y = 5^x$ and reflect about $y = x$ to get $y = \log_5 x$ -OR-
   (b) Make a chart of values for $y = 5^x$. Switch the x and y values to get a chart for sketching $y = \log 5^{x}$.

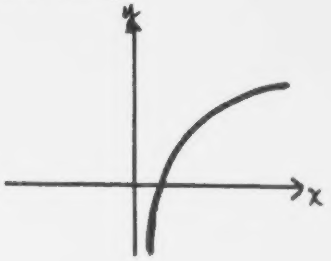
2. (a) (0, 1)
   (b) (1, 0)

3. $x = 3^y$ and $y = \log_3 x$ are the same

4. $y = 3^x$ and $y = \log_3 x$ are inverses
   (Also accept: $y = 3^x$ and $x = 3^y$ are inverses.)

*5a) $(-8)^{1/2}$, $(-4)^{3/4}$, etc. would involve even roots of negative numbers which are imaginary. Since we are dealing with graphs now, we take values of a yielding REAL numbers only.

*5b)



In general, logarithmic graphs lie in QI and QIV only as shown in this diagram. (This came from the reflection of $y = a^x$ about $y = x$). Thus, the graph is NEVER in QII or QIII disallowing negative values of x.
(Similarly: For $y = a^x$, $y > 0$ its inverse $y = \log_a x$ has the property $x > 0$.)

*6) 9 ($y = 2^x$ and $y = \log_2 x$ are inverses, so performing one of the functions on "9" and then performing the other function on the result will yield "9".)

SUMMARY:

1. Ask, "How are exponential and logarithmic functions related?"
2. Ask, "What should you ask yourself if you have to find $\log_3 243$?"
3. Ask, "What synonym do you think of when you hear the word logarithm?"
4. What logarithmic form is equivalent to $x = b^y$?

LESSON: 18

AIM: How do we find $\log_b a$?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Change between the exponential form and the logarithmic form of an equation.
2. Find the log of a number, $\log_b a$ (where a is a power of b).

VOCABULARY: Logarithm

MOTIVATION:

1. Ask "What question must you ask yourself in order to evaluate $\log_2 8$?"
2. Evaluate $\log_2 8$.

DEVELOPMENT:

1. The above leads to:
   In order to find $\log_2 8$ we must ask ourselves
   "2 to what power gives 8?" (Answer : 3)
2. Ask, "How did we define $y = \log_b x$ yesterday?"
   Elicit: If $b > 0$, $b \neq 1$, then $y = \log_b x$ means $b^y = x$.
3. An equation involving powers can be written in TWO forms: For example; the numbers 3, 4 and 64 can be used to write an equation in either of two forms:
   EXPONENTIAL FORM: $4^3 = 64$ $(\text{base})^{\text{exponent}} = \text{power}$
   LOGARITHMIC FORM: $\log_4 64 = 3$ $\log_{\text{base}}\text{power} = \text{exponent}$
   - (Remember - a logarithm is an exponent.)
4. Solve the following equation by changing to exponential form:
   $\log_x 125 = 3$ means $x^3 = 125$; $x = 5$
5. Do Applications 1a) through 1h), 2 a-e, 3, 4, 5, 6.

6. **Tell the students that in our course, the most common base for logarithms is base 10. The tables we will use are base 10 logarithm tables. The first person to use base 10 logarithms was the mathematician Henry Briggs (1561 -1630). These logarithms are called "Common Logarithms."**
7. **Do Application 7.**
8. **Show that in application example 7 we got negatives each time since $\log_b x$ is negative if $0 < x < 1$.**

APPLICATIONS:

1) Complete the following Table

	EXPONENTIAL FORM		LOGARITHMIC FORM
a)	$2^5 = 32$	a)	
b)	$3^2 = 9$	b)	
c)		c)	$\log_5 25 = 2$
d)		d)	$\log_{10} 10 = 1$
e)		e)	$\log_5 1 = 0$
f)	$3^{-1} = \frac{1}{3}$	f)	
g)	$2^{-3} = \frac{1}{8}$	g)	
h)	$a^x = b$	h)	

2) Find each of the following logarithms
   a) $\log_{10} 100$
   b) $\log_4 16$
   c) $\log_2 16$
   d) $\log_{16} 2$
   e) $\log_{1000} 1$

3) If $b > 0$, find
   a) $\log_b b$
   b) $\log_b 1$
   c) $\log_b (\frac{1}{b})$

4) Given $\log_b a = c$

Which of the numbers a, b, c can be

a) positive

b) negative

c) 0

d) Which MUST be positive?

5) Find

a) $\log_{10} 10$

b) $\log_{10} 100$

c) $\log_{10} 10{,}000$

d) $\log_{10} 100{,}000{,}000$

e) Make a generalization as a result of ex. 5 a-d.

6) When is $\log_b x$ negative?

7) Find

a) $\log_2(\frac{1}{2})$

b) $\log_3(\frac{1}{9})$

c) $\log_4(\frac{1}{16})$

d) $\log_2(\frac{1}{16})$

SOLUTIONS TO APPLICATION PROBLEMS:

1) a) $\log_2 32 = 5$

b) $\log_3 9 = 2$

c) $5^2 = 25$

d) $10^1 = 10$

e) $5^0 = 1$

f) $\log_3(\frac{1}{3}) = -1$

g) $\log_2(\frac{1}{8}) = -3$

h) $\log_a b = x$

2) a) 2

b) 2

c) 4

d) 1/4

e) 0

3) a) 1
   b) 0
   c) -1

4) a) a, b, c
   b) c
   c) c
   d) a, b

5) a) 1
   b) 2
   c) 4
   d) 8
   e) If a number $\log_{10}x$ is to be found and x consists of a "one" followed by m zeros, then $\log_{10}x = m$ or if $x = 10^n$, then $\log_{10}x = n$

6. If $0 < x < 1$

7) a) -1
   b) -2
   c) -2
   d) -4

SUMMARY:

1. Ask, "What exponential equation is equivalent to: $A = \log_C D$?"
   Elicit: $C^A = D$

2. Correct the errors:
   a) $\log_7 2 = 49$
   b) $\log_5 2 = 32$

LESSON: 19

AIM: How do we use common logarithm tables?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Use a log table to find the logarithm of a number.
2. Use a log table to find a number, given its logarithm.
3. Find the characteristic and mantissa of a logarithm.

VOCABULARY: Characteristic, mantissa

MOTIVATION:

Distribute the following table:

POWERS OF 2

$2^0 = 1$
$2^1 = 2$
$2^2 = 4$
$2^3 = 8$
$2^4 = 16$
$2^5 = 32$
$2^6 = 6$
$2^7 = 128$
$2^8 = 256$
$2^9 = 512$
$2^{10} = 1024$
$2^{11} = 2048$
$2^{12} = 4096$
$2^{13} = 8192$
$2^{14} = 16384$
$2^{15} = 32768$
$2^{16} = 65536$

Use the table to help find:

(1) $\sqrt{65536}$ (2) $\sqrt[3]{4096}$

DEVELOPMENT:

1. Elicit in the above examples

$$\sqrt{65536} = (2^{16})^{1/2} = 2^8 = 256$$

$$\sqrt[3]{4096} = (2^{12})^{1/3} = 2^4 = 16$$

2. Ask, "Which of the following examples can be simplified using the above table?"

   a) $\frac{16384}{1024}$

   b) 16384 + 65536

   c) 2048 - 512

   d) 256 x 128

   (Elicit: Tables of Powers can help simplify multiplication and division.)

*3. Let us include $2^{.5} = 1.4142$ in the table. Ask class to find:

   a) $2^{1.5}$

   b) $2^{3.5}$

   c) $2^{-.5}$

   Elicit:

   a) $2^{1.5} = 2^{1}.2^{.5} = 2\ (1.4142) = 2.8284$

   b) $2^{3.5} = 2^{3}.2^{.5} = 8(1.4142) = 11.3136$

   c) $2^{-.5} = 2^{-1}(2^{15}) = \frac{1}{2}\ (1.4142) = .7071$

   Note how we had to manipulate when the power is negative.

4. Ask the class to describe the type of exponent if a number between 1 and 10 is written as a power of 10. Elicit a number between 1 and 10 = 10^{decimal} since $1 = 10^{0}$ and $10 = 10^{1}$ and numbers between 0 and 1 may be written as a decimal.

5. Distribute a table of common logarithms. Tell your students that they should ignore the title. Tell them, further, that this table is really a table of powers of 10. The numbers from 1.00 to 9.99 appear (the whole number part and tenths appear down the left column, the hundredths appear part across the top row). The table can be used as follows: ex: 4.52 should be located. The number appearing in this position is "6551". This means that $4.52 = 10^{0.6551}$. (Note that the decimal is understood automatically since it must be inserted for each example.)

6. Do Applications 1a, b, c, d, e.

7. Ask, "Can we use these tables to rewrite a) 356 b) .257 c) 13600 as powers of 10?"

Elicit:

a) $356 = 10^2(3.56) = 10^2\ (10^{0.5514}) = 10^{2.5514}$

(Note: We must first write the number in scientific notation.)

b) $25.7 = 10^1(2.57) = 10^1(10^{0.4099}) = 10^{1.4099}$

c) $13600 = 10^4(1.3600) = 10^4(10^{0.1335}) = 10^{4.1335}$

8. Summarize some of the above results:

Exponential Form	Logarithmic Form
$10^{0.6551} = 4.52$	$\log_{10} 4.52 = 0.6551$
$10^{0.8189} = 6.59$	$\log_{10} 6.59 = 0.8189$
$10^{2.5514} = 356$	$\log_{10} 356 = 2.5514$
$10^{4.1335} = 13600$	$\log_{10} 13600 = 4.1335$

9. Ask, "Can you use the table to find $\log_{10}$ 51.3?" (1.7101) (Note, if we write log 51.3 (without a base), base 10 is understood). Ask, "Can you use the table to find log 6280?" Also ask, "What additional difficulty do we have in this example?" Place the following definitions on the board:

DEFINITIONS:

1) The decimal part of a common logarithm is called the "mantissa." **The log table is really a table of mantissas.**
2) The integer part of a common logarithm is called the "characteristic." Identify the mantissa and characteristic of each logarithm in development #7 and #8. Elicit the characteristic is the power of 10 when the number is put in scientific notation. Note that the tables give us the mantissas only.

10. Do Applications 2 a-e, 3.

APPLICATIONS:

1. Use your table to find x
   a) $10^x = 4.59$
   b) $10^x = 2.21$
   c) $10^x = 9.90$
   d) $10^x = 6.59$
   e) $x = 10^{0.7810}$

2. Use your table to find
   a) log 6.970
   b) log 385000
   c) log 49.00
   d) log .000134
   e) log .000000640

3. Ask, "How can we get the characteristic of a number by just looking at the number?

SOLUTIONS TO APPLICATION PROBLEMS:

1a) 0.6618
b) 0.3444
c) 0.9956
d) 0.8189
e) 6.04

2a) 0.8432
b) 5.5855
c) 1.6902
d) -4 + 0.1271
e) -7 + 0.8062

3) The characteristic is the power of 10 when the number is written in scientific notation.

SUMMARY:

Tell class, "Use your table to rewrite 46.8 as a power of 10 and show the corresponding logarithmic equation."

($10^{1.6702} = 46.8$; log 46.8 = 1.6702)

LESSON: 20

AIM: How do we find the logarithm of a product?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Find the product of two numbers by using logarithms.
2. Use linear interpolation in conjunction with the log tables.
3. Apply the laws of logarithms concerning products ($\log_a xy$).

VOCABULARY: linear interpolation, antilogarithm

MOTIVATION:

Using the logarithm table as a Table of powers of 10, find (34.5) (6.78)

DEVELOPMENT:

1. Elicit from your students:
   Let x = (34.5) (6.78)

$$x = (10^{1.5378})(10^{.8312}) = 10^{2.3690}$$
$$x = 10^{2.3690}$$
$$x = 10^{2}(10^{.3690})$$
$$= 10^{2}(2.34)$$
$$= 234$$

   Ask, "What number must we look up in the table?" Elicit that the characteristic is 2 and the mantissa is .3690. We can round off the result since .3690 does NOT appear in the table. The table gives $2.34 = 10^{.3690}$.

2. Note that we used logarithms to help find product. More specifically:

   a) $x = (10^{1.5378}) \quad (10^{.8312}) \quad = \quad 10^{2.3690}$

$$1.5378 + .8312 = 2.3690$$
$$\log 34.5 + \log 6.78 = \log x$$
$$\log x = 2.3690$$

b) To get x, we used the table "in reverse". We are finding the number given its logarithm. This is called finding the "antilog" of a number. Thus
$x = \text{antilog } 2.3690$.

c) Note that we **ADDED** two logarithms in order to get the log of the product. In general, we write the theorem:
Theorem: If $x = mn$, then $\log_B x = \log_B m + \log_B n$
i.e., log product = sum of the logs of its factors.

3. **(For "average classes"):** Consider another example:

Rewriting these with logarithms:

Let $m = 3^5$ $\qquad$ $\log_3 m = 5$
$n = 3^6$ $\qquad$ $\log_3 n = 6$
$x = m \cdot n = 3^{11}$ $\qquad$ $\log_3 x = 11$
Thus $x = 3^{11}$

Note: $\log_3 x = \log_3 m + \log_3 n$. This further demonstrates our theorem.
(This is just another way of writing the law of exponents.)

4. (For "above average classes"): We prove this theorem

Let $\log_B m = a$ $\qquad$ This means that $\qquad$ $m = B^a$
$\log_B n = b$ $\qquad$ $n = B^b$
$x = mn = B^a B^b$
$x = B^{a+b}$ Rewrite as logs
$\log_B mn = a+b$ $\quad$ or $\quad$ $\log_B mn = \log_B m + \log_B m$

5. Do Application 1. Stress
   - a) Estimate,
   - b) Log equation
   - c) Work column
   - d) Antilog
   - e) Does answer agree with estimate?
   - f) To get the final result, students MUST interpolate. It would be beneficial to have homework problems reviewing this technique for a few nights previous t this lesson.

6. Do Applications 2 and 3.

APPLICATIONS:

1. Use logarithms to find (8.64) (97.5) to the nearest tenth.

2. Use the fact that $\log_{10}2 = .3010$, $\log_{10}3 = .4771$ and $\log_{10} 10 = 1$ to find:
   a) log 4
   b) log 8
   c) log 6
   d) log 12
   e) log 54
   f) log 5.4

3. When you found log 4, you obtained .6020. Your calculator gave .6021 to four places! Explain the discrepancy.

SOLUTIONS TO APPLICATIONS PROBLEMS:

1. Let x = (8.64)(97.5); Estimate: x = (9)(100) = 900

log x = log 8.64 + log 97.5

$$\begin{array}{rlcl} & \log\ 8.64 & = & 0.9365 \\ + & \underline{\log\ 97.5} & \underline{=} & \underline{1.9890} \\ & \log\ x & = & 2.9255 \end{array}$$

$\log x = 2.9255$ or $x = 10^2(10^{.9255})$

$$\begin{array}{ll} \underline{\text{number}} & \underline{\text{mantissa}} \end{array}$$

$$10\left[\begin{array}{l} a\left[\begin{array}{ll} 8420 & 9253 \\ 8420+a & 9255 \end{array}\right]2 \\ \\ \quad 8430 \qquad\quad 9258 \end{array}\right]5$$

$\frac{a}{10} = \frac{2}{5}$

$a = 4$

$x = 10^2(10^{.9255})$

$x = 10^2(8.424)$

$x = 842.4$

2. a) $\log 4 = \log (2\cdot 2) = \log 2 + \log 2 = 2(.3010) = .6020$
   b) $\log 8 - \log (4\cdot 2) = .6020 + .3010 = .9030$
   c) $\log 6 = \log 2 + \log 3 = .3010 + .4771 = .7781$
   d) $\log 12 = \log (4\cdot 3) = .6020 + .4771 = 1.0791$
   e) $\log 54 = \log (6\cdot 9) = .7781 + 2(.4771) = 1.7323$
   f) $\log 5.4 = \log 54(\frac{1}{10}) = \log 54 + \log .1 = 1.7323$
   $+ (-1) = .7323$

3. Log 2 = .3010 is AN APPROXIMATION. Rounding log 2 to one more place gives .30103.
   Log 4 = 2(.30103) = .60206 which rounds to .6021.
   Note that there are some "rounding errors" in using this technique; however, they are usually small.

SUMMARY:

Ask, "How can we use logarithms, to find the log of a product?" Elicit the log of a product is equal to the sum of the logs of its factors.

Ask, "If calculators are so readily available, why should we need to be able to calculate with logarithms?"

Elicit that the logarithmic function is extemely important in higher mathematics since many applications involve logarithms. It is more important to know such properties as $\log_B xy = \log_B x + \log_B y$ as a property of this special function, rather than being able to do many messy calculations using logarithms.

LESSON: 21

AIM: How do we find the logarithm of a quotient?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Find the quotient of two numbers using logarithms.
2. Apply the law of logarithms concerning quotients.

VOCABULARY: Antilogarithm

MOTIVATION:

If $x = \frac{72.9}{3.46}$, find x using your log table.

DEVELOPMENT:

1. In the above $x = \frac{10^{1.8627}}{10^{0.5391}} = 10^{1.3236} \approx 21.1$

   Ask, "Can you formulate a theorem from the above results?" (Since log 72.9 = 1.8627, log 3.46 = 0.5391 we used log x = log 72.9 - log 3.46.)

   THEOREM: If $x = \frac{m}{n}$, then $\log_B x = \log_B m - \log_B n$

   i.e., The log of a quotient = (log numerator) - (log denominator).

2. **For "average classes".**

   Let $m = 4^{20}$ $\qquad \log_4 m = 20$

   $n = 4^6$ $\qquad - \log_4 n = 6$

   $x = \frac{m}{n} = \frac{4^{20}}{4^6} = 4^{14}$ $\qquad \log_4(\frac{m}{n}) = 14$

   $\therefore \log_4(\frac{m}{n}) = \log_4 m - \log_4 n$

   This further demonstrates our theorem.

For "above average classes".

Proof: Let $a = \log_B m$ This means $B^a = m$

$c = \log_B n$ $B^c = n$

$$x = \frac{m}{n} = \frac{B^a}{B^c} = B^{a-c}$$

$$\frac{m}{n} = B^{a-c} \quad \log_B \left(\frac{m}{n}\right) = a-c$$

or: $$\log_B \left(\frac{m}{n}\right) = \log_B m - \log_B n$$

3. Do Applications 1-4.
4. Do Application 5. Be sure to stress (as in previous lesson):
   a) Estimate
   b) Log equation
   c) Work column
   d) Antilogarithm
   e) Does answer agree with estimate?

   Note that to simplify the calculations:
   log 46.8 = 1.6702 or 11.6702 -10
   log .057 = -2 + .7559 or 8.7559 -10
   Often in subtractions of logarithms, we avoid difficulties by writing a negative characteristic as 8 -10, 7 -10, 9 -10 etc.

5. Do (difficult) Application 6. (If time permits.) Be sure to separate the numerator and denominator calculations. Do Application 7.

APPLICATIONS

1. Rewrite $\log_3 10 - \log_3 2$ more simply
2. Simplify $\log_b \frac{1}{a}$
3. Simplify $\log_{10} \frac{x}{10}$
4. Solve for x: $\log_4 x - \log_4 5 = \log_4 3 + \log_4 \pi$
5. Find $\frac{46.8}{.057}$ using logarithms
6. Use logarithms to find

$$x = \frac{(8.64)\ (97.5)}{(468)\ (.579)}$$

7. If log 5 = a

log 2 = b

log 6 = c

Express the following in terms of a, b and c

a) log 4

b) log 2.5

c) log 3

*d) log 15

*e) log 45

SOLUTIONS TO APPLICATION PROBLEMS:

1. $\log_3 \frac{10}{2} = \log_3 5$
2. $\log_b 1 - \log_b a = 0 - \log_b a = -\log_b a$
3. $\log_{10} x - \log_{10} 10 = \log_{10} x - 1$
4. $\frac{x}{5} = 3\pi \quad x = 15\pi$
5. Estimate $\frac{46.8}{.057} \approx \frac{46800}{60} \approx 780$

Let $x = \frac{46.8}{.057}$

log x = log 46.8 -log.057

$$\begin{array}{rrl} & \log 46.8 & = 11.6702-10 \\ - & \underline{\log .057} & \underline{= 8.7559-10} \\ & \log x & = 2.9143 \\ & x & = 10^2(10^{.9143}) \\ & x & = 100(8.21) \\ & x & = 821 \end{array}$$

(Compare with estimate.)

6. Estimate: $\frac{9 \times 100}{500 \times .6} = \frac{0900}{300} \approx 3$

log quotient = (log numerator) - (log denominator)

log x = (log 8.64 + log 9.75) - (log 468 + log.579)

(Be sure to emphasize the use of parentheses.)

To find log Num.

$$\begin{array}{rl} & \log 8.64 = 0.9365 \\ + & \underline{\log 97.5 = 1.9890} \\ & \log \text{Num} = 2.9255 \\ - & \underline{\log \text{Den} = 2.4329} \\ & \log x = .4926 \\ & x = 10^{.4926} \\ & x = 3.11 \end{array}$$

(Compare with estimate.)

To find log Denom.

$$\begin{array}{rl} & \log 468 = 2.6702 \\ + & \underline{\log .579 = 9.7627-10} \\ & \log \text{Den} = 12.4329-10 \\ & \log \text{Den} = 2.4329 \end{array}$$

7. a) 2b
   b) a-b
   c) c-b
 * d) a+c-b
 * e) a+2c-2b

SUMMARY:

Rewrite another way when possible:

a) log ab	(log a + logb)
b) (loga) (logb)	(Cannot simplify)
c) $\log \frac{a}{b}$	(log a - logb)
d) $\frac{\log a}{\log b}$	(Cannot simplify (at this stage!)

How would you find the log(quotient)?

LESSON: 22

AIM: How do we find the logarithm of a power or root? How do we use logarithms to simplify a difficult calculation problem?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Apply the law of logarithms to an expression involving exponents ($\log_a x^r$).
2. Apply the law of logarithms concerning exponents.
3. Use logarithms in order to find the value of a variable given by a formula.
4. Write a logarithmic equation for a given complicated formula.

MOTIVATION:

If $x = (3.45)\cdot(3.45)\cdot(3.45)\cdot(3.45)$, use logarithms to find x.

DEVELOPMENT:

1. In the above, $\log x = 4 \log 3.45$, $\log x = 4(0.5378) = 2.1512$ so $x = \text{antilog } 2.1512 \approx 142$

   Ask, "Can you formulate a theorem generalizing the above results?" Elicit:

THEOREM: If $x = m^r$, where r is ANY real number, then $\log_B x = r\log_B m$.

i.e., $\log(\text{number})^{\text{power}} = \text{power}(\log \text{number})$

2. **For "average classes"**

   We demonstrate this theorem with an example where r is a whole number: Let $x = m^{10}$ or $x = \underbrace{m\cdot m\cdot m\ldots m}_{\text{10 factors of m}}$

   $\log x = \underbrace{\log m + \log m + \ldots + \log m}_{\text{10 factors of log m}}$ (Using the product rule.)

   Thus $\log x = 10\log m$

   For "above average classes"

   We prove the theorem: Given: $x = m^r$
   Prove: $\log_B x = r\log_B m$

We will consider the right hand side

Let $\log_B m = a$ Thus $m = B^a$

In order to get x, we raise m to the r^{th} power

$x = m^r = (B^a)^r = B^{ar}$. So $x = B^{ar}$. Rewriting this in terms of logs we get $\log_B x = ar = r \log_B m$ Q.E.D.

3. Note: This theorem is true if the exponent r is ANY real number. For example, if $x = \sqrt{a}$ then $x = a^{1/2}$ and $\log_B x = \frac{1}{2} \log_B a$.

4. Do Applications 1 (a) - (f).

5. Let us apply all recent work to a complicated problem

Find the value of $\dfrac{8.7 \times \sqrt[3]{0.0595}}{(0.861)^2}$

Solution

a) Let $x = \dfrac{8.7 \times \sqrt[3]{0.0595}}{(.861)^2}$

b) Estimate: $x \approx \dfrac{9 \times .4}{.81} = \dfrac{.40}{.09}$

$x \approx 4.4$

c) $\log x = \underbrace{\log 8.7 + \tfrac{1}{3}\log 0.0595}_{\text{log numerator}} - \underbrace{2\log .861}_{\text{log denominator}}$

Annotations

a) We need an EQUATION in order to use logarithms

b) At the end we want to see if the results make sense

Note $\sqrt[3]{.064} = .4$

c) We need a log equation

d)

$$\begin{array}{rl} & \log 8.7 = 0.9395 \\ + & \underline{\tfrac{1}{3} \log 0.0595 = 9.5915-10} \\ & \log \text{Num} = 10.5310-10 \\ - & \underline{\log \text{Den} = \overset{-}{\oplus}9.8700\overset{+}{\ominus}10} \\ & \log x = .6610 \\ & x = 10^{.6610} \\ & x = 4.58 \end{array}$$

(Compare with estimate.)

Work

$\log .0595 = 8.7745-10 = 28.7745-30$

(We need to divide by 3. Thus it is better to have -30 in the problem than -10.)

$\frac{1}{3} \log .0595 = 9.5915 -10$

$\log 0.861 = 9.9350 -10$

$2 \log 0.861 = 19.8700 -20 =$

$= 9.8700 -10$

Be sure to emphasize how we made "adjustments" by changing 8.7745 -10 to 28.7745 -30.

6. Do Applications 2 and 3**.

APPLICATIONS:

1. Let log 2 = a and log 3 = b. Express each of the following in terms of a and b:

a) log 4
b) log 9
c) log 6
d) log $\sqrt{2}$
e) log $\sqrt{6}$
f) log 54
g) log 40.5
h) log $\sqrt[3]{12}$

Write the log equation you would use to find

2. a) $\sqrt{256}$
b) $\sqrt[3]{\dfrac{3V}{4\pi}}$
c) $r = \sqrt{\dfrac{(1.2)(465)}{16.3 \sin 72°}}$

**3. Suppose you needed $\log_5 7$ and only had a log 10 table. How would you do it?

SOLUTION TO APPLICATION PROBLEMS:

1. a) $\log 2^2 = 2 \log 2 = 2a$
b) $\log 9 = \log 3^2 = 2 \log 3 = 2b$
c) $\log 6 = \log 2 + \log 3 = a+b$
d) $\log 2^{1/2} = \frac{1}{2} \log 2 = \frac{1}{2}a$
e) $\log 6^{1/2} = \frac{1}{2} \log 6 = \frac{1}{2}(a+b)$
f) $\log 54 = \log (3^3 \cdot 2) = 3b+a$
g) $\log 40.5 = \log \frac{81}{2} = \log \frac{3^4}{2} = 4b-a$
h) $\log (2^2 \cdot 3)^{1/3} = \frac{1}{3}(2a+b)$

2. a) Let $x = 256^{1/2}$

$\log x = \frac{1}{2} \log 256$

b) Let $x = \sqrt[3]{\frac{3V}{4}} = \frac{3V}{4\pi}^{1/3}$

$\log x = \frac{1}{3}\left[(\log 3 + \log V) - (\log 4 + \log \pi)\right]$

c) $\log r = \frac{1}{2}\left[(\log 1.2 + \log 465) - (\log 16.3 + \log \sin 72°)\right]$

**3. Let $x = \log_5 7$

$5^x = 7$

$x \log_{10} 5 = \log_{10} 7$

$x = \frac{\log_{10} 7}{\log_{10} 5} = \frac{.8451}{.6990} \approx 1.21$

SUMMARY:

Ask: "Explain the difference between $(\log_{10} 5)^4$ and $\log_{10} 5^4$."

(Elicit that the first gives the fourth power of $\log_{10} 5$
The second is the log of 5^4 OR $4 \log_{10} 5$.)

Ask: "How would you find the $\log(\text{number})^{\text{power}}$?"

LESSON: 23

AIM: How do we solve exponential and logarithmic equations?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Solve exponential equations.
2. Solve logarithmic equations.

*3. Find a log of a number with base other than 10.

VOCABULARY: Exponential equation, logarithmic equation

MOTIVATION:

Solve for x: 1) $4^x = 32$

2) $4^x = 50$

DEVELOPMENT:

1. Ask, "What makes example (2) more difficult than example (1)?" Elicit that in the first exponential equation, both sides can be expressed as a power of 2, while this is NOT so in (2).
   Elicit the solution in (1): $(2^2)^x = 2^5$ $2^{2x} = 2^5$
   $x = 2.5$.

2. Ask for ideas on how to solve the second equation. If no one suggests using logarithms, ask how they would solve $x^4 = 50$ ($x = \sqrt[4]{50}$). Here it was easy to isolate x. In (2) we isolate x by eliminating the exponent! Here logarithms are handy!

$$4^x = 50 \; ; \; \log_{10} 4^x = \log_{10} 50$$

$$x \log 4 = \log 50$$

$$x = \frac{\log 50}{\log 4} = \frac{1.6990}{0.6021}$$

   To get x use long division to obtain 2.82

3. Do Applications 1 and 2.

4. Elicit that the equations considered so far today are "exponential equations." Similar to exponential equations are "logarithmic equations" (equations containing logarithms). For example: $\log_x 3 = 15$ (What base must log 3 be taken to in order to yield 15?) Elicit that we have BASE 10 tables available. Thus we must rewrite in exponential form, THEN take base 10 logs. Thus: $x^{15} = 3$; $15 \log_{10} x = \log_{10} 3$

   $\log_{10} x = \frac{1}{15} (.4771) = .0318$; Thus $x = 10^{.0318} = 1.08$

5. Do Applications 3 and 4.

*6. Do Application 5.

**7. Do Application 6.

APPLICATIONS:

1. Solve for x to the nearest hundredth
   $x^5 = 50$
2. Solve for x to the nearest hundredth
   $5^x = 50$
3. Solve for x to three significant digits
   a) $\log_7 3x = 2$
   b) $\log_7 2 = 3x$
   c) $\log_{3x} 7 = 2$
4. Solve to three significant digits WITHOUT USING the quadratic formula: $(x+3)^2 = 32$

*5. Find $\log_4 9$ to three significant places

**6. Suppose you only have a base 10 log table and need $\log_b a$. Find a formula that would enable you to find this needed logarithm.

SOLUTIONS TO APPLICATION PROBLEMS:

1. $5 \log x = \log 50$

   $\log x = \frac{1}{5} \log 50 = \frac{1}{5} (1.6990) = .03398$

   $x = 2.19$

2. $x \log 5 = \log 50$; $x = \frac{\log 50}{\log 5} = \frac{1.6990}{0.6990}$

   $x = 2.43$

3. a) $7^2 = 3x$; $x = \frac{49}{3} = 16.3$

   b) $7^{3x} = 2$; $3x \log 7 = \log 2$

   $x = \frac{.3010}{3(.8451)} = .119$

   c) $(3^x)^2 = 7$; $3^{2x} = 7$

   $2x \log 3 = \log 7$

   $x = \frac{.8451}{2(.4771)} = .886$

4. $2 \log (x+3) = \log 32$

   $\log (x+3) = \frac{1.5051}{2} = .7526$ - Now take antilogarithms

   $x+3 = 5.66$

   $x = 2.66$

   Note - a good original estimate would be 2.5

*5. Let $x = \log_4 9$

   $4^x = 9$

   $x = \frac{\log 9}{\log 4} = \frac{.9542}{.6021} = 1.58$

**6. Let $x = \log_b a$; $a = bx$; $x \log_{10} b = \log_{10} a$

   $x = \frac{\log_{10} a}{\log_{10} b}$

   Note: This generalizes to any base $c > 0$: $\log_b a = \frac{\log_c a}{\log_c b}$

SUMMARY:

Ask "When do we use logarithms to help solve an equation?"

Elicit: For
- a) Exponential equations that have different bases on both sides
- b) Equations leading to roots of a number
- c) Equations with messy multiplications and/or divisions
- d) An equation involving a log with an unknown base.
- e) To find a log of a number with a base other than 10.

LESSON: 24*

* Note this lesson may be done with above average students.

AIM: How do we apply logarithms to practical problems involving growth?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Apply logarithms to interest problems.
2. Apply logarithms to population problems.

VOCABULARY: Interest, compound interest, principal, growth

MOTIVATION:

If $5000 is deposited in a bank at 8% compounded quarterly, how much money will there be at the end of 5 years?

DEVELOPMENT:

1. Ask, "What difficulties are there in solving the above problem?" Elicit - "interest earns interest!" Let us simplify the problem first, then come back to it.

2. Consider starting with $1.00 with a 4% rate compounded annually. Develop the following chart:

At the end of	Amount (A)
1st year	$A = \$1.00 + 0.4 = \$1.04$
2nd year	$1.04 + 0.4\ (1.04) =$ $(1.04)\ (1 + .04) =$ $A = (1.04)^2 = \$1.08$
3rd year	$(1.04)^2 + .04\ (1.04)^2$ $= (1.04)^2\ (1 + .04)$ $A = = (1.04)^3 = \$1.12$
Elicit: t^{th} year	$A = (1.04)^t$

3. Let us consider the results with other rates. Ask, "Had the bank given 5% annually, what formula could we use to calculate A?" Elicit: $A = (1.05)^t$. Develop the following chart:

Rate	Amount (A) after t^{th} year
5% = .05	$A = (1.05)^t$
6% = .06	$A = (1.06)^t$
12% = .12	$A = (1.12)^t$
r (Where r is already written as a decimal.)	$A = (1 + r)^t$

4. Let's consider a 4% rate SEMIANNUALLY

After	Amount	Amount for rate of r
One half year	$A = (1 + .02)^1$	$A = (1 + \frac{r}{2})^1$
One year	$A = (1.02)^2$	$A = (1 + \frac{r}{2})^2$
$1\frac{1}{2}$ years	$A = (1.02)^3$	$A = (1 + \frac{r}{2})^3$
two years	$A = (1.02)^4$	$A = (1 + \frac{r}{2})^4$
t years	$A = (1.02)^{2t}$	$A = (1 + \frac{r}{2})^{2t}$

Ask, "How would the formula change if the interest were compounded

a) quarterly? $A = (1 + \frac{r}{4})^{4t}$

b) monthly?" $A = (1 + \frac{r}{12})^{12t}$

c) daily?" $A = (1 + \frac{r}{365})^{365t}$

d) n times yearly?" $A = (1 + \frac{r}{n})^{nt}$

5. The above describes what happens for 1 dollar. Ask, "What do you get after t years, n times yearly for $2?

$$A = 2(1 + \frac{r}{n})^{nt}$$

In general, we start with a principal of $P. Ask, "What formula predicts the amount?" Elicit the "growth formula":

$$\mathbf{A = P(1 + \frac{r}{n})^{nt}}$$

6. We return to the original problem:

$$A = 5000\ (1 + \frac{.08}{4})^{4\cdot 5}$$

$$A = 5000\ (1.02)^{20}$$

$\log A = \log 5000 + 20 \log 1.02$; $\log 5000 = 3.6990$
$\log A = 3.8710$ $\qquad \log 1.02 = 0.0086$
$A = \$7430.00$ $\qquad 20 \log 1.02 = 0.1720$
at the end of 5 years $\qquad \log 5000 + 20 \log 1.02 = 3.8710$

7. Do Application 1.

8. Ask "Why is population growth similar to interest in a bank?" (Yield - The more people you have, the more the increase will be). Apply the growth formula to Application 2, and if time, Applications 3 and 4.

APPLICATIONS:

1. Benjamin Franklin left $5000 to the City of Philadelphia in 1791 to be redeemed in 1891 using an ANNUAL rate of 5%. Explain why his will could not be carried out!

2. The world's population rate is approximately 2% yearly. It has been constant for many years.
   In 1975 the population was 4 billion. Predict the population in:
   a) 1990
   b) 2000

3. How long does it take population (money) to double at a 2% rate (compounded annually)?

4. Redo 3 for a 10% annual rate.

SOLUTIONS TO APPLICATION PROBLEMS:

1. $A = 5000\,(1 + .05)^{100} = 5000(1.05)^{100}$

$\log A = \log 5000 + 100 \log 1.05$

$\log A = 5.8190$

$A = \$659{,}000$

$\log 5000 = 3.6990$

$\log 1.05 = 0.0212$

$\underline{100 \log 1.0 = 2.12}$

$\log 5000 + 100 \log 1.0 = 5.8190$

$659,000 was considered a great sum in 1891. Thus his estate was given only $100,000.

2. a) $A = 4(1.02)^{15}$

$\log A = \log 4 + 15 \log 1.02$

$\log A = .7311$

$A = 5.38$ Billion

$\log 4 = .6021$

$\log 1.02 = .0086$

$\underline{15 \log 1.02 = 0.1290}$

$\log 4 + 15 \log 1.02 = .7311$

b) $A = 4(1.02)^{25}$

$\log A = \log 4 + 25 \log 1.02$

$\log A = .8171$

$A = 6.56$ Billion

$25 \log 1.02 = .2150$

$\underline{+ \ \log 4 = .6021}$

$\log A = .8171$

3. $2P = P\,(1.02)^{x}$

$2 = (1.02)^{x}$

$x \log 1.02 = \log 2$

$x = \frac{.3010}{.0086} = 35$ years

4. $2P = P\,(1.1)^{x}$

$2 = 1.1^{x}$

$x = \frac{\log 2}{\log 1.1} = \frac{.3010}{.0414} = 7.27$ years

SUMMARY:

Ask, "How can we use logarithms to help predict our principal in the bank in future years?"

Elicit: $A = P\left(1 + \frac{r}{n}\right)^{nt}$

This formula uses a power of a number slightly more than 1. We therefore can use the product and power rules for logarithms.

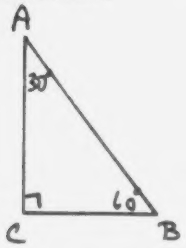
LESSON: 25

AIM: In any triangle, how does the ratio of the sides compare to the ratio of the measures of the sines of the angles opposite these sides?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Derive the formula: $A = \frac{1}{2}ab \sin C$ from the formula $A = \frac{1}{2}bh$ and apply the formula.
2. Derive the law of sines from the formula $A = \frac{1}{2}ab \sin C$, and write the formula in different ways.
3. Use the law of sines to find an unknown side of a triangle if measures are given for A.A.S. or A.S.A.

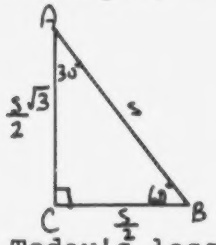
MOTIVATION.



"List the sides of this triangle in order, from shortest to longest." ($\overline{BC}$, $\overline{AC}$, $\overline{AB}$)

Ask, "What is the ratio of $\overline{AC}$ to $\overline{BC}$?"

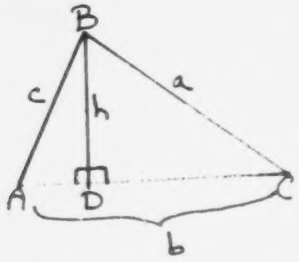
The class should recall the sides of the 30° - 60° Right Triangle, and obtain $\sqrt{3}$:



$$\frac{\frac{s}{2}\sqrt{3}}{\frac{s}{2}} = \sqrt{3}$$

Today's lesson will enable us to find the ratio of the lengths of the sides of any triangle whose angles are known.

DEVELOPMENT:



Ask, "What formula gives us the area of a triangle?" ($\frac{1}{2}bh$)

In right triangle ABD, find sin $\measuredangle$ A. ($\sin \measuredangle A = \frac{h}{c}$)

In right triangle CBD, find sin $\measuredangle$ C. ($\sin \measuredangle C = \frac{h}{a}$)

Solving both equations for h, we obtain $h = c \sin \measuredangle A$
$h = a \sin \measuredangle C$.

Therefore, $A = \frac{1}{2}bh = \frac{1}{2}bc \sin \measuredangle A$
and $A = \frac{1}{2}bh = \frac{1}{2}ba \sin \measuredangle C$.

Express these formulas verbally. (The area of a triangle is $\frac{1}{2}$ the product of the lengths of two sides and the sine of the included angle.)

Now, since the area is unchanged, $\frac{1}{2}bc \sin \measuredangle A = \frac{1}{2}ba \sin \measuredangle C$.

From the above, obtain algebraically $\frac{\sin \measuredangle A}{a} = \frac{\sin \measuredangle B}{b}$; $\frac{\sin \measuredangle A}{\sin \measuredangle B} = \frac{a}{b}$ and their reciprocals. By drawing an altitude from C to AB, the above can be extended to $\frac{\sin \measuredangle A}{a} = \frac{\sin \measuredangle B}{b} = \frac{\sin \measuredangle C}{c}$, etc.

What do these formulas tell us about the ratio of sides of a triangle? **(The ratio of the sides of a triangle is equal to the ratio of the sines of the angles opposite those sides.)** This is known as the Law of Sines. Note the triangle need not be a right triangle, and that (as in any proportion), if we know 3 of 4 quantities, we may solve for the fourth. Ask: "Under what circumstances would we use the Law of Sines?" Elicit if we are given two angles and a side, and asked to find another side. Why are two sides of an angle sufficient to do this? (Both A.S.A and A.A.S. are sufficient conditions for congruence.)

APPLICATIONS:

1. Find the area of a triangle with sides of 6 and 7 and included angle of 60°.

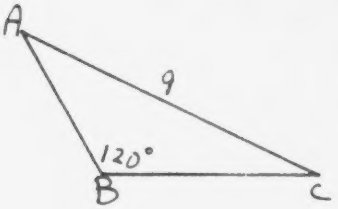
2. In Δ ABC. a = 7.0, b = 20, and sin ∡ A = 0.21. Find the value of sin ∡ B.

3. In Δ ABC, A = 30°, B = 45°, b = $20\sqrt{2}$. Find a

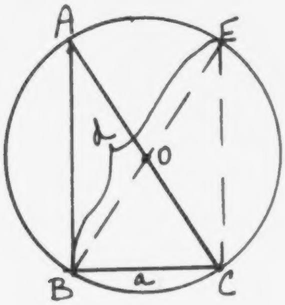
4. In triangle ABC, angle A is twice angle B. What is the ratio of side a to side b?

5. A triangle whose area is $10cm^2$ has two of its sides of length 6 cm and 5 cm respectively. What is the angle between these 2 sides?

6. In Δ ABC, AC = 9, m ∡ B = 120, m ∡ C = 35. If the perimeter of ABC is 19, find the lengths of the remaining sides.



**7. Use the diagram shown to determine $\frac{a}{\sin A}$.



SOLUTIONS TO APPLICATION PROBLEMS:

1. A = $\frac{1}{2}$ab sin C = $(\frac{1}{2})(6)(7)\sin 60° = 21(.8660)$ 18.2 OR $(\frac{1}{2})(6)(7) \cdot \frac{3}{2} = \frac{21}{2}$ 3
(Discuss the difference between an exact answer and an approximate answer.)

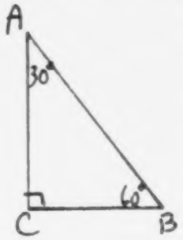
2. $\frac{a}{\sin\measuredangle A} = \frac{b}{\sin\measuredangle B}$; $\frac{\sin\measuredangle B}{20} = \frac{0.21}{7.0}$

   $\sin\measuredangle B = 0.6$

3. $\frac{\sin 30^\circ}{a} = \frac{\sin 45^\circ}{20\sqrt{2}}$; $a = \frac{20\sqrt{2}\cdot\frac{1}{2}}{\frac{\sqrt{2}}{2}} = \frac{10\sqrt{2}}{\frac{\sqrt{2}}{2}} = 20$

4. $A = 2B$, $\frac{a}{b} = \frac{\sin\measuredangle A}{\sin\measuredangle B} = \frac{\sin 2B}{\sin B}$ which cannot be simplified further at this point.

   However, if for instance $m\measuredangle A = 60$ then $m\measuredangle B = 30$ and $\frac{a}{b} = \frac{\sin 60^\circ}{\sin 30^\circ} = \frac{\sqrt{3}/2}{\frac{1}{2}} = \sqrt{3}$; note the ratio is not 2:1.



5. $A = \frac{1}{2}ab\sin\measuredangle C$

   $10 = \frac{1}{2}(6)(5)\sin\measuredangle C$

   $\sin\measuredangle C = 0.667$; $\measuredangle C = 41^\circ 50'$ or $138^\circ 10'$

6. $\frac{9}{\sin 120^\circ} = \frac{9}{0.8660} = 10.39$

   $\frac{AB}{\sin 35^\circ} = 10.39 = \frac{AB}{0.5736}$ $AB \approx 6.0$

   $\therefore BC = 19-(9+6) = 4$

** 7. Since $\measuredangle A = \angle E$, $\sin\measuredangle A = \sin\measuredangle E$

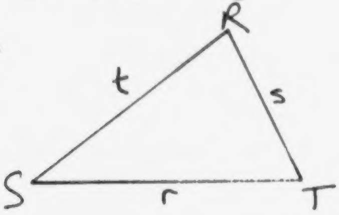
   Also, $m\measuredangle BCE = 90$, $\sin\measuredangle A = \sin\measuredangle E = \frac{a}{d}$

   $\therefore d = \frac{a}{\sin\measuredangle A}$

SUMMARY:

1. State the Law of Sines for a, b, $\sin\measuredangle A$, $\sin\measuredangle B$ in four different ways.

2.



   Write the formula for the Law of Sines for the triangle shown.

   $\left(\frac{r}{\sin\measuredangle R} = \frac{s}{\sin\measuredangle S} = \frac{t}{\sin\measuredangle T}\right)$

3. Under what circumtances is the Law of Sines used?

LESSON: 26

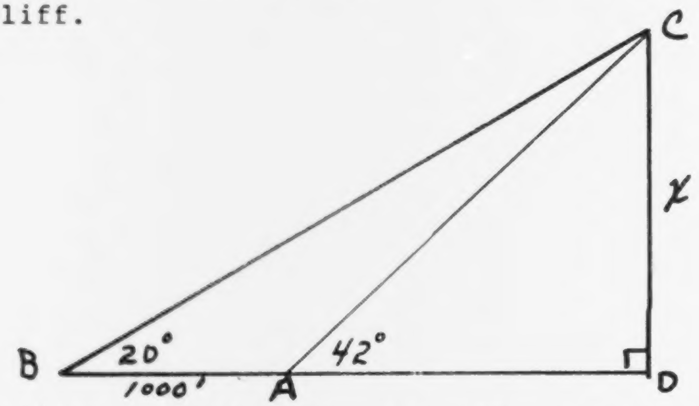
AIM: How is the Law of Sines used to find the length of a side of a triangle?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Determine when the law of sines can be applied to find unknown sides of a triangle, and then apply the most convenient form of the equation.
2. Use a log-trig. table in computing the solution of a law of sines problem.

MOTIVATION:

The angle of elevation of a cliff is 42° from point A and 20° from point B which is 1000 feet from A. Find the height of the cliff.



First consider triangle ABC. Ask, "How can we use the measurement of 1000 feet?" (We can find ∡BCA and then use the Law of Sines to find $\overline{AC}$.)

"If we find $\overline{AC}$, how can this help us to find $\overline{CD}$?" ($\sin 42° = \frac{x}{AC}$)

DEVELOPMENT:

The preceeding problem should be solved by the class algebraically and optionally, using logorithms.

$$m\angle BCA = m\angle CAD - m\angle ABC$$
$$= 42 - 20 = 22$$
$$\frac{AC}{\sin 20°} = \frac{1000}{\sin 22°}$$

$$\frac{AC}{0.3420} = \frac{1000}{0.3746}$$

$$\sin 42° = \frac{x}{913}$$

$$0.6691 = \frac{x}{913}$$

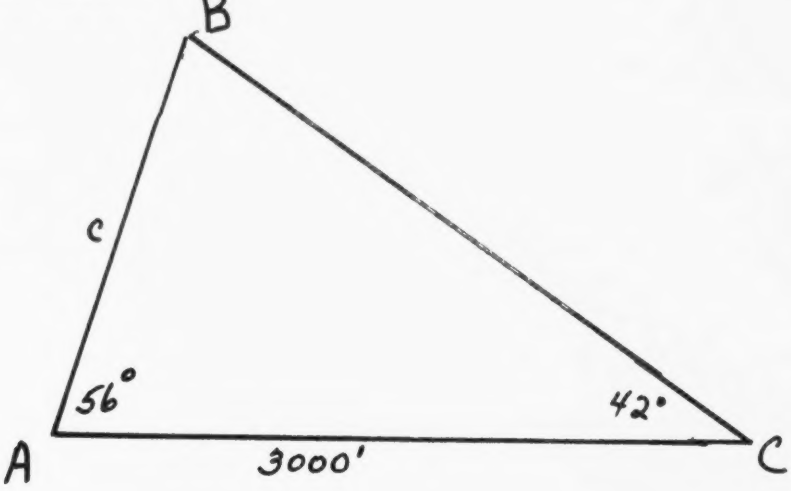
$$x = 611 \text{ feet}$$

$$\begin{aligned} \log AC &= \log 1000 + \log \sin 20° - \log \sin 22° \\ &= 3 + 9.5341 - 10 - (9.5736 - 10) \\ &= 2.9605 \\ AC &= \text{antilog } 2.9605 \approx 913 \text{ feet} \end{aligned}$$

(If this method is taught, students should first be asked to find sin 20° = 0.3420 and then to find the logarithm of that number. The tables for logarithms of trigonometric functions should then be introduced. The characteristic of 9-10 should be stressed, and the absence of -10 from the log-trig. tables noted. These tables can be seen as a time-saving device eliminating the necessity for using two different tables.)

APPLICATIONS:

1. In triangle ABC, m∡A = 52°40', m∡B = 26°10', side c = 125 feet. Find side a to the nearest foot.
2. In triangle ABC, m∡A = 85°40', m∡C = 56°20' and a = 12. Find the length of side c to the nearest integer.
3. A weather balloon is directly over a 3000 foot airstrip. The angles of elevation from two points on the ground, A and C, as shown in the diagram are 56° and 42°. Find the distance from the balloon (B) to point A (to the nearest hundred feet).



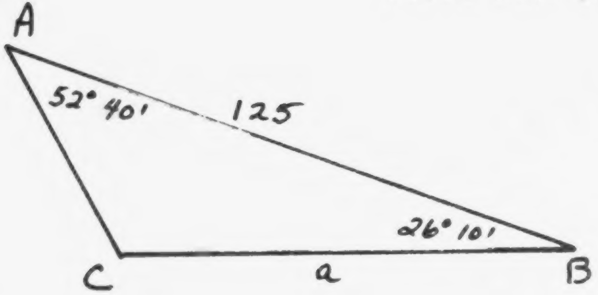
4. Two girls from a surveying class establish a base line AB on a level field. The girl at point A is 50 feet from the girl at the point B. Each one sights a stake at point C. The girl at A measures ∢CAB to be 78°40'. The girl at B measures ∢CBA to be 92°50'. Find, to the nearest foot, the distance from A to C.

5. A man at one point on the street finds that the angle of elevation of the top of a tower is 29°50'. After walking toward the tower for 200 feet in a straight line, he finds that at the second point the angle of elevation of the top of the tower is 65°20'. What is the height of the tower to the nearest foot?

SOLUTIONS TO APPLICATION PROBLEMS:

1.

First find ∢C = 180°-(26°10'+52°40')
= 101°10'



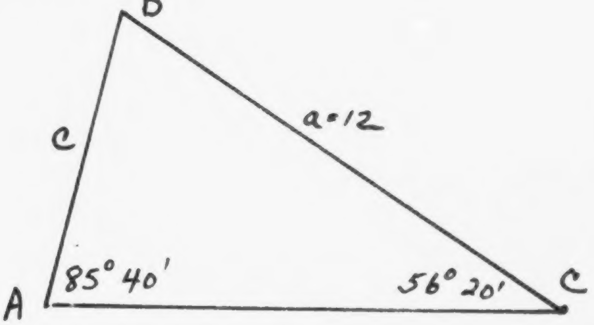
$$\frac{a}{\sin 52°40'} = \frac{125}{\sin 101°10'}$$

sin 101°10' = sin 78°50'
= 0.9811

sin 52°40' = 0.7951

$$a = \frac{125(.7951)}{.9811}$$ 101 feet

2.



$$\frac{c}{\sin 56°20'} = \frac{12}{\sin 85°40'}$$

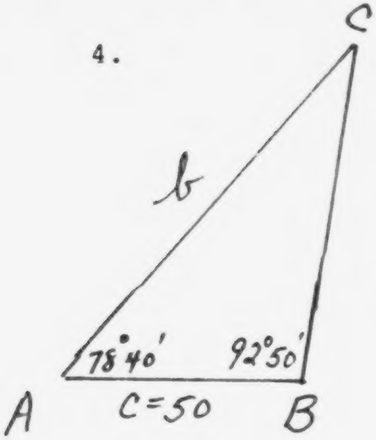
$$c = \frac{12 \sin 56°20'}{\sin 85°40'} = \frac{12(0.8323}{0.9971} \approx 10$$

3. $m\angle B = 180-(56+42) = 82$

$$\frac{c}{\sin 42^\circ} = \frac{3000}{\sin 82^\circ}$$

$$c = \frac{3000 \sin 42^\circ}{\sin 82^\circ} = \frac{3000(0.6691)}{0.9903} \approx 2000 \text{ feet}$$

4.



First find $m\angle C = 8°30'$

$$\frac{b}{\sin 92°50'} = \frac{50}{\sin 8°30'}$$

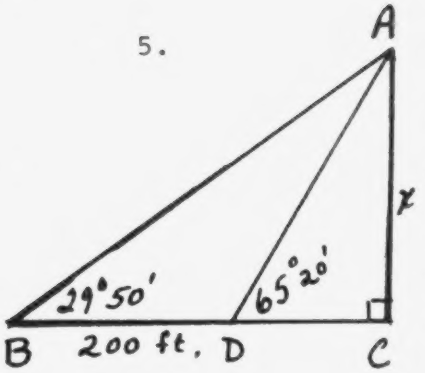
Since $\sin 92°50' = \sin 87°10'$

$= 0.9988$

and $\sin 8°30' = 0.1478$

$$b = \frac{50(0.9988)}{0.1478} = 338 \text{ ft (to nearest foot)}$$

5.



$\angle BAD = 65°20' - 29°50' = 35°30'$

$$\frac{AD}{\sin 29°50'} = \frac{200}{\sin 35°30'}$$

$$AD = \frac{200(0.4975)}{0.5807} \approx 171 \text{ ft.}$$ (to nearest foot)

(Logs may be convenient for this problem and #5.)

$\sin 65°20' = \frac{AC}{171}$; $AC = 171(.9088)$

$AC = 155$ ft (to nearest foot)

SUMMARY:

Ask, "How have we used the Law of Sines thus far?" (To find a side of a triangle when two angles and one side are given.)

"Considering that the formula involves two sides and two angles, what other possible applications may be possible?" (Finding an angle when two sides and one angle are given.)

LESSON: 27

AIM: How can the Law of Sines be used to find the measure of an angle if SSA is given?

PERFORMANCE OBJECTIVES: The student will be able to...

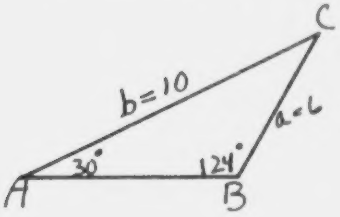
1. Determine the number of triangles possible if SSA is known.
2. Solve for all possible values of the unknown angle using the Law of Sines.

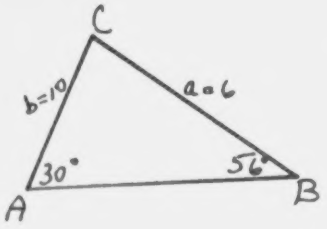
VOCABULARY: ambiguous

MOTIVATION:

A) If sin ∡B = 0.8333, find ∡B.
From the tables, students will find m∡B = 56. Ask "How would it be possible for ∡B to be obtuse?" (∡B could measure 124°.)

B) Now pose the problem: In triangle ABC, m∡A = 30, a = 6, b = 10. Find ∡B. Using the Law of Sines, $\frac{\sin\angle B}{b} = \frac{\sin\angle A}{a}$ or $\sin\angle B = \frac{b \sin \angle A}{a} = \frac{10 \sin 30°}{6} = \frac{5}{6} = 0.8333$
Referring back to the first question, we find ∡B to be 56° or 124°.

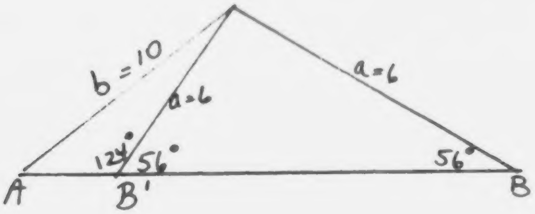




This is called the ambiguous case of the Law of Sines. It results when S.S.A. is given, since these measures do not necessarily produce congruent triangles.

DEVELOPMENT:

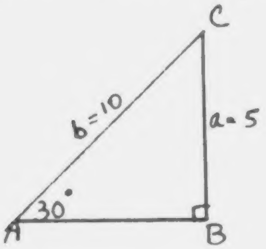
Now consider the diagram "combining" the two shown above.



Suppose, as before, m ∡A = 30, b = 10, but now a = 5. Applying the Law of Sines, we obtain:

$$\sin \measuredangle B = \frac{b \sin \measuredangle A}{a} = \frac{10 \sin 30^\circ}{5} = \frac{5}{5} = 1$$

Find ∡B. (B can only be 90°.)

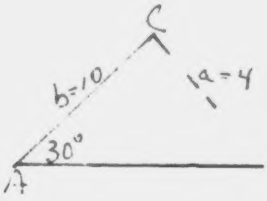


Note the above equation $b \sin \measuredangle A = a$ which states that $\sin \measuredangle A = \frac{a}{b}$. Thus a is the opposite leg of a right triangle, or the altitude to $\overline{AB}$ in the triangle ABC.

How does the length of "a" compare to b sin ∡A in the example which resulted in the ambiguous case?
(b sin ∡A = 5, a = 6 thus a > b sin ∡A.)

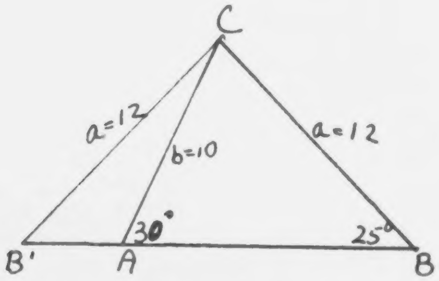
A board compass is useful here in demonstrating the relation of side "a" to the altitude from C.

Next let a = 4 and obtain $\sin \measuredangle B = \frac{b \sin \measuredangle A}{a} = \frac{5}{4}$ which is not possible. "What does this mean geometrically?" (The side is shorter than the altitude, and thus no triangle is possible.)



Let a = 12. $\sin \measuredangle B = \frac{b \sin \measuredangle A}{a} = \frac{10 \sin 30°}{12} = \frac{5}{12} = 0.4167$

$\therefore$ m∡B = 25 or 155. The latter is not possible since m∡A = 30. Also the supplement of ∡ A rather than ∡A would be an angle of the triangle. Thus one triangle is possible. If m∡A $\geq$ 90, "a" must be the longest side of the triangle. If this is the case, there is one solution; otherwise, (if "a" is not the longest side) there are no solutions.

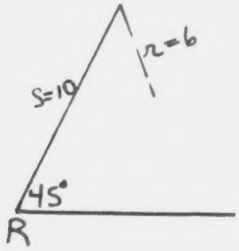


APPLICATIONS:

1. How many different triangles can be constructed if m∡R = 45°, r = 6, s = 10?
2. In triangle LMN, m∡L = 55°, n = 8, ℓ = 7. Find m∡N to the nearest degree.
3. In Δ ABC, a = 10, b = 11, sin ∡A = 0.8660. Find m∡B to the nearest ten minutes.
4. If, in Δ ABC, m∡A = 30, a = $\sqrt{5}$ and b = 4, then ∡B
   - a) may be either obtuse or acute
   - b) must be obtuse, only
   - c) must be acute, only
   - d) may be a right angle

SOLUTIONS TO APPLICATION PROBLEMS:

1.



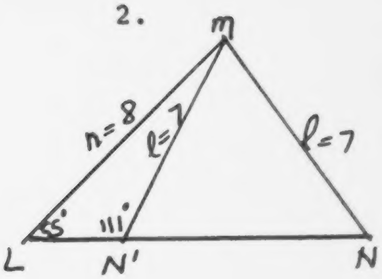
$$\frac{\sin \measuredangle S}{s} = \frac{\sin \measuredangle R}{r} =$$

$$\sin \measuredangle S = \frac{s \sin \measuredangle R}{r} = \frac{10 \sin 45°}{6}$$

= 1.1785

NO SOLUTION

2.



$$\frac{\sin \measuredangle N}{n} = \frac{\sin \measuredangle L}{\ell} =$$

$$\sin \measuredangle N = \frac{n \sin \measuredangle L}{\ell} = \frac{8 \sin 55°}{7} = 0.9362$$

Also $n > \ell$, therefore $\measuredangle N > \measuredangle L$.

$m\measuredangle N = 69°$ or $111°$

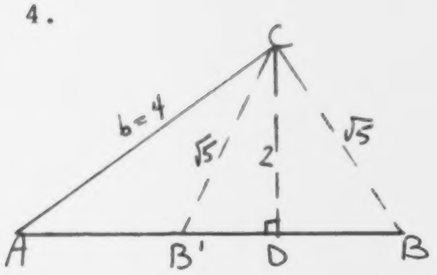
TWO SOLUTIONS ARE POSSIBLE.

3. $\frac{\sin \measuredangle B}{11} = \frac{0.8660}{10}$

$\sin \measuredangle B = 0.9526$

$m \measuredangle B \approx 72°20'$ or $180°-72°20' = 107°40'$

4.



First find CD=2 (30°-60°rt.Δ)
since $2 < \sqrt{5} < 4$, two triangles are possible and choice (a) is correct.

or

$\frac{\sin A}{a} = \frac{\sin B}{b}$;

$\sin B = \frac{4 \sin 30}{\sqrt{5}} = \frac{2}{\sqrt{5}} = .8944$

$a < b$, so $\measuredangle A < \measuredangle B$
and $\measuredangle B$ may be 63° or 117°

SUMMARY:

Ask, "How can the Law of Sines be used to find an angle if $\measuredangle A$, a and b are given?

Elicit: Use the Law of Sines to find $\sin \measuredangle B$.

a) If $\sin \measuredangle B > 1$, no solution is possible.

b) If $\sin \measuredangle B = 1$, one solution (a right triangle) is obtained.

c) If $0 < \sin \measuredangle B < 1$, two solutions are obtained if $a < b$ (the ambiguous case); one solution is obtained if $a \geq b$.

LESSON: 28

AIM: How can we solve trigonometric problems using the Law of Sines?

PERFORMANCE OBJECTIVES: The student will be able to...

*1. Solve problems related to astronomy using the Law of Sines.

*2. Apply the concept of navigational bearings.

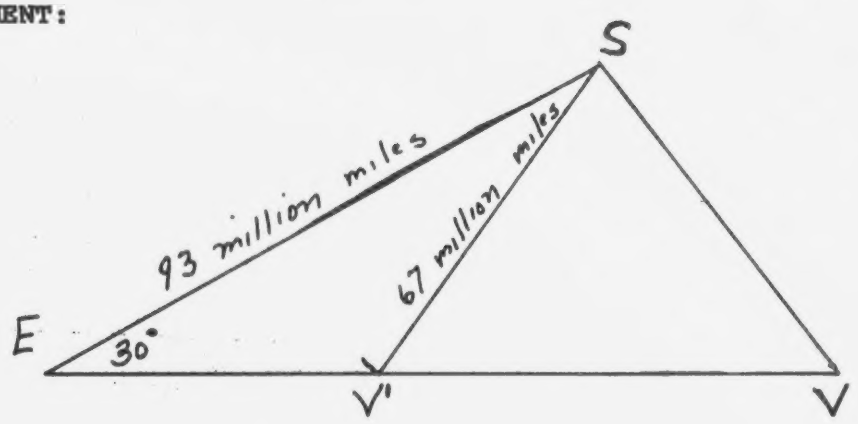
*3. Solve navigational and other problems using the Law of Sines.

VOCABULARY: navigational, nautical bearings

MOTIVATION:

The distance of the Earth, E, from the sun, S is approximately 93,000,000 miles, and distance of Venus, V, from the sun is approximately 67,000,000 miles. At a certain time angle SEV is found to be 30°. How far is Venus from the Earth?

DEVELOPMENT:



Have the class observe that v sin E = 93 sin 30° = 46.5, and thus v sin E < e < v .˙. there are two solutions:

$\text{Sin } V = \frac{v \sin E}{e} = \frac{46.5}{67} = 0.6940$ (approx.)

$V \approx 44°$ or $136°$

114

Let us tabulate our data in 2 columns.

$v = 93$	$v = 93$
$e = 67$	$e = 67$
$\angle V = 44°$	$\measuredangle EV'S = 136°$
$\angle E = 30°$	$\angle E = 30°$
$\therefore \angle S_1 = 106°$	$\therefore \angle S_2 = 14°$
$EV = \dfrac{v \sin S_1}{\sin v}$	$EV' = \dfrac{v \sin S_2}{\sin \measuredangle EV'S}$
$EV = \dfrac{93 \sin 106°}{\sin 44°}$	$EV' = \dfrac{93 \sin 14°}{\sin 136°}$
$EV = 129$ (approx.)	$EV' = 32$ (approx.)

Therefore, Venus is either 32 million or 128 million miles from the Earth at the time the measurements were made. Discuss the physical significance of the two solutions - Venus may be in two different positions in its orbit around the sun and still satisfy the conditions of the problem. Additional facts are needed to determine which of the two distances is the correct one at any particular time.

APPLICATIONS:

1. Two vessels leave the same port at the same time. The faster vessel follows a course of 022 at 15 knots (nautical miles per hour). The slower vessel follows a course of 037. At the end of two hours the vessels are 10 nautical miles apart. What is the speed of the slower vessel?

   Note: Nautical bearings are measured clockwise from North, using three significant digits. Thus, 022 means N22°E, while 122 means S58°E.

2. Trapezoid ABCD has $\overline{BC} \parallel \overline{AD}$ and $\overline{CD} \perp \overline{AD}$. $\overline{CB}$ is extended through B to P and diagonal AC is drawn. If the measure of angle BCA is 29°40', the measure of angle PBA is 41°20' and BC = 1.8, find CD to the nearest tenth.

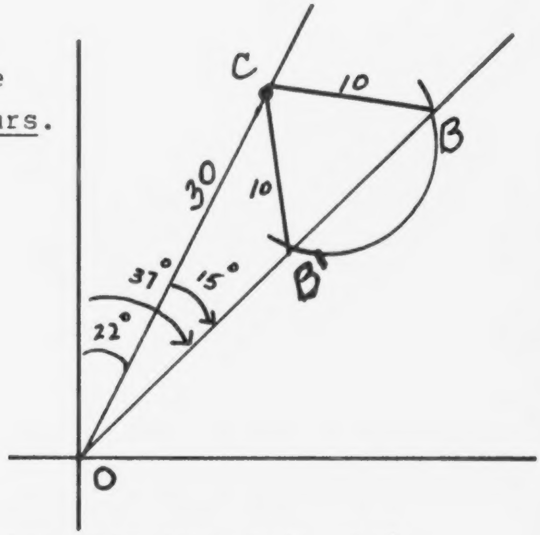
3. A is 300 miles N 40° E of B. C is due south of A and due east of B. Find the distance from A to C.

4. A ship at A is 80 miles due west of a ship at B. An SOS is received by both ships showing the bearing of a distressed ship to be N 72° E from A and N 65° W from B. Find to the nearest mile the distance from the distressed ship to the closer of the other two ships.

5. From two points 250 yards apart on a horizontal (straight) road running directly toward the launch pad, the angles of elevation of the top of a rocket measure 44° and 28°. Find the height of the rocket to the nearest ten yards.

SOLUTIONS TO APPLICATION PROBLEMS:

1. $\triangle$s OCB and OCB' represent the possible situation after 2 hours.



Notice that this is the ambiguous case once more, and again there are two solutions since 30 sin 15° < 10 < 30.

$$\text{Sin } B = \frac{30 \sin 15^\circ}{10} = 0.7764$$

$$\therefore B = 51^\circ \text{ or } 129^\circ$$

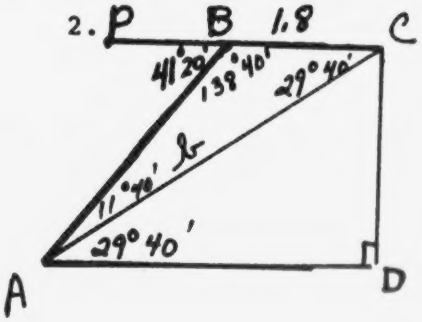
$$\therefore \angle OCB = 114^\circ, \angle OCB' = 36^\circ$$

However, in this case the class should observe that only one solution meets the conditions of the problem; since if we use OCB = 114°, OB > OC. But if B is the point where the slower vessel will be at the end of 2 hours, OC should be greater than OB, therefore, we use OB'C = 129° and OCB' = 36°.

$$OB' = \frac{30 \sin 36^\circ}{\sin 129^\circ} = 22.7$$

And the rate of the slower boat is approximately 6 knots.

2.


P
B
1.8
C
41°20'
138°40'
29°40'
b
11°40'
29°40'
A
D

First find AC

$$\frac{b}{\sin 138^\circ 40'} = \frac{1.8}{\sin 11^\circ 40'}$$

$$\sin 138^\circ 40' = \sin 41^\circ 20'$$

$$b = \frac{1.8 \sin 41^\circ 20'}{\sin 11^\circ 40'}$$

$$\log b = \log 1.8 + \log \sin 41^\circ 20' - \log \sin 11^\circ 40'$$

$$= 0.2553 + 9.8198 - 10 - (9.3058 - 10) = 0.7693$$

$b = 5.9$ to nearest tenth

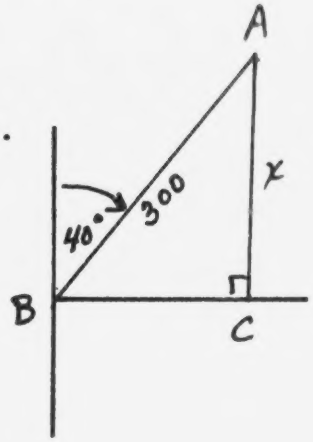
$$\sin \angle CAD = \frac{CD}{5.9}$$

$$CD = 5.9 \sin 29^\circ 40'$$

$$= 5.9(0.4950) = 2.9205$$

$CD = 2.9$ (to the nearest tenth)

3.

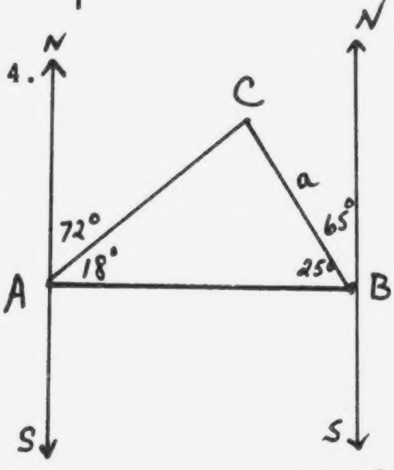


Since $m\angle ABC = 50$, $\sin 50^\circ = \frac{x}{300}$

$$0.7660 = \frac{x}{300}$$

$x \approx 230$ ft.

4.


N
N
C
a
72°
18°
65°
25°
A
B
S
S

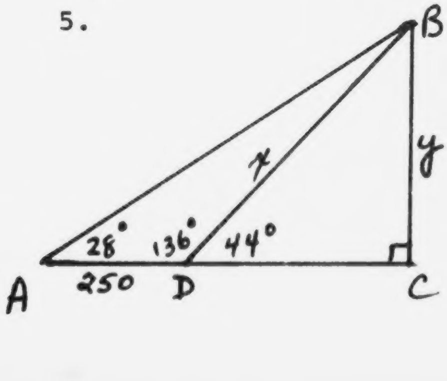
First, we find $m\angle C = 137$

$$\frac{80}{\sin 137^\circ} = \frac{a}{\sin 18^\circ}$$

$$a = \frac{80(0.3090)}{0.6820} = 36 \text{ miles}$$

(to nearest mile)

5.


B
x
y
28°
136°
44°
A
250
D
C

First find $m\angle ADB = 136$, $m\angle ABD = 16$

$$\frac{x}{\sin 28^\circ} = \frac{250}{\sin 16^\circ}$$

$$x = \frac{250(0.4695)}{0.2756} = 426$$

$$\sin 44^\circ = \frac{y}{426}$$

$y = 426(0.6947) = 296$ yds.

(to nearest yard)

SUMMARY:

Ask the class when the Law of Sines can be used.

1) If two angles and one side were given (A.S.A. or S.A.A.).

or

2) If two sides and an angle opposite one of the sides are given (ambiguous case) (S.S.A).

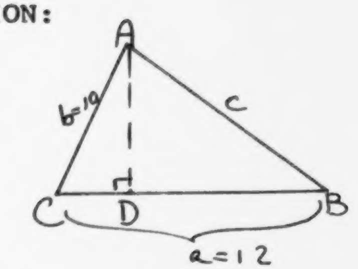
LESSON: 29

AIM: How can we find the length of a side or the measure of an angle of a triangle if only SAS is known?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Derive the Law of Cosines and write the formula in various ways.
2. Apply the Law of Cosines and find, for instance, a if b, c and ∡ A are known.

MOTIVATION:



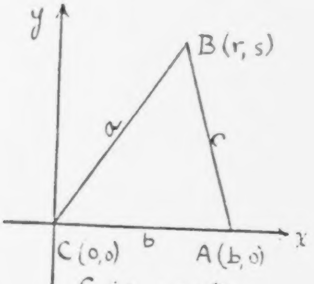
Draw diagram without altitude. Find length of side c.

The class should recognize that the Law of Sines will not be applicable here since no angle and its opposite side are known. Students may suggest drawing altitude $\overline{AD}$ and then determining $\overline{AD}$, $\overline{CD}$, $\overline{DB}$. Angle B can then be determined and used to find $\overline{AB}$.

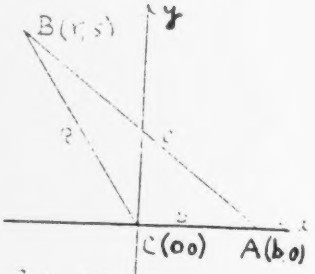
In today's lesson we will learn a more direct method to find a side when S.A.S. is known.

DEVELOPMENT:

Introduce a set of coordinate axes so that one of the angles, say angle C is in standard position.



C is acute



C is obtuse

The students should determine the coordinates of the vertices of triangle ABC. Note that B is a point somewhere in Quad. I or II and can be represented by (r,s).

Remind the students that we are interested in obtaining a formula for side c. They should suggest the use of the distance formula.

$$c^2 = (r - b)^2 + s^2 = r^2 - 2br + b^2 + s^2$$

Since the above formula contains r and s, which are not sides of the given triangle ABC, it is not in a desirable form. The problem is to obtain a formula containing only the parts of the triangle ABC. Students should then see that $r^2 + s^2 = a^2$ and that $r = a \text{ Cos } C$ (whether C is acute or obtuse).

Therefore, by substitution, the class should obtain the formula

$$\mathbf{c^2 = a^2 + b^2 - 2ab \text{ Cos } C}$$

After realizing that the angle placed in standard position was arbitrarily chosen, students should be able to state similar formulas for a^2 and b^2; their derivation can be assigned for homework. These three relations are known as **the Law of Cosines.** They hold for all triangles and the class should observe that if $\measuredangle C = 90°$, then $c^2 = a^2 + b^2 -2ab \cos 90°$ or $c^2 = a^2 + b^2$. Therefore, the Pythagorean theorem is a special case of the Law of Cosines.

APPLICATIONS:

1. In triangle RST, $s = 13$, $t = 15$, $m \measuredangle R = 50°$. Find side r.
2. In CDE, $c=8$, $e=12$, $M \measuredangle D = 120$. Find side d.
3 In ABC, $a=5$, $b=12$, $m \measuredangle C = 90$. Find side c.
4. A local airline does not offer a direct connection from city A to city B. Rather, the flight travels 70 miles from city A to city C and then 100 miles from city C to city B. If the angle ACB between the two legs of the flight is 100^0, find to the nearest mile the distance between A and B.

5. At 2 PM on a certain day, a ship sailed from a port ~~on a~~ course 71° 20 (N 71° 20 E) at a speed of 15 miles per hour. At 3 PM, a second ship left the same port on a course 131° 10 (S 48° 50 E) at a speed of 20 miles per hour. Find, to the nearest mile, the distance between the ships at 5 PM.

SOLUTIONS TO APPLICATION PROBLEMS:

1. $r^2 = s^2 + t^2 - 2st \cos \measuredangle R$

$= 13^2 + 15^2 - 2(13)(15)\cos 50°$

$= 169 + 225 - 390(0.6428)$ (Stress order of operation.)

$= 394 - 250.7$ (Note $r^2 < s^2 + t^2$ when $\measuredangle R$ is acute.)

$= 143.3$

$r = \sqrt{143.3} \approx 12$

2. $d^2 = c^2 + e^2 - 2ce \cos \measuredangle D$

$= 8^2 + 12^2 - 2(8)(12)\cos 120^0$

since $\cos 120° = - \cos 60° = \frac{-1}{2}$

$d^2 = 64 + 144 - 2\ (8)(12)\ (\frac{-1}{2})$

$= 208 + 96 = 304$ (Note that $d^2 > c^2 + e^2$ since $\measuredangle D$ is obtuse.)

$d = \sqrt{304} \approx 17.4$

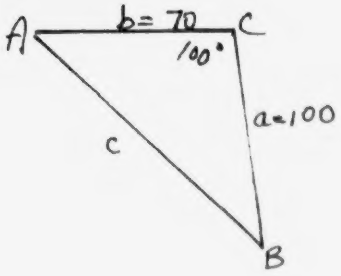
3. $c^2 = a^2 + b^2 - 2ab \cos \measuredangle C.$

$= 5^2 + 12^2 - 2(5)(12)\cos 90°$

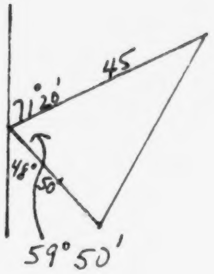
since $\cos 90° = 0$,

$c^2 = 5^2 + 12^2 = 169$

$c = \sqrt{169} = 13$ (Note that the Law of Cosines reduces to the Pythagorean Theorem when the angle is a right angle.)

4. 

$$c^2 = 100^2 + 70^2 - 2(100)(70)\cos 100°$$
$$\cos 100^{c} = -\cos 80° = -0.1736$$
$$c^2 = 10{,}000 + 4{,}900 - 14{,}000(-0.1736)$$
$$= 17{,}330.4$$
$$c = \sqrt{17{,}330.4} = 132 \text{ miles}$$
(to the nearest mile)

5. 

$$x^2 = 40^2 + 45^2 - 2(40)(45)\cos 59°50'$$
$$= 1600 + 2025 - 3600(0.5025)$$
$$= 1816$$
$$x = \sqrt{1816} = 43 \text{ miles}$$
(to the nearest mile)

SUMMARY:

1. Ask "Under what conditions do we apply the Law of Cosines?" (When S.A.S. is given, and we wish to find the third side of a triangle.)
2. Ask "Under what other conditions is it possible to use of Law of Cosines?" (When S.S.S. is given and we wish to find any angle.)

LESSON: 30

AIM: How can we use the Law of Cosines to find the angles of a triangle when S.S.S. is known?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Determine if a triangle is acute, obtuse or right, using the Law of Cosines and the lengths of the three sides.
2. Find the measure of an angle, given the 3 sides, using the Law of Cosines.

MOTIVATION:

The sides of a triangle are 10,24,26. Find the measure of the largest angle. The class should recall that the largest angle of a triangle is opposite the largest side; therefore, if the class represents the sides as follows:
$a = 10$, $b = 24$, $c = 26$, the problem becomes that of finding angle C.

$$26^2 = 10^2 + 24^2 - 2(10)(24)\cos \angle C$$
$$\cos \angle C = \frac{0}{400} = 0$$

Therefore, the measure of angle C is 90. The triangle is a triangle similar to a 5, 12, 13 right triangle.

Note that S.S.S. is a sufficient reason for proving conguence; in other words a unique triangle is formed if the lengths of the three sides are given.

DEVELOPMENT:

Now change the largest side in the above triangle to a length of 30. Ask "How would this affect the angle opposite this side?" (It should increase in measure).

$$30^2 = 10^2 + 24^2 - 2\ (10)(24)\ \cos \angle C$$
$$900 = 576 - 480 \cos \angle C$$
$$\text{or } \cos \angle C = \frac{-324}{480} = -0.675 \qquad \text{thus } m \angle C \ \pi\ 132.$$

NOTE that when $c^2 > a^2 + b^2$, $\angle C$ is obtuse.

Now make the sides 10, 24, 25

Solving for $\measuredangle C$,

$$25^2 = 10^2 + 24^2 - 2(10(24)\cos \measuredangle C$$

$$625 = 676 - 480\cos \measuredangle C$$

$$\text{or } \cos \measuredangle C = \frac{51}{480} = 0.1063$$

$$\text{thus } m \measuredangle C \approx 84$$

Here, $c^2 < a^2 + b^2$ and $\measuredangle C$ is acute.

APPLICATIONS:

1. The sides of a triangle are 9, 12, 13. Find the measure of the smallest angle.
2. The sides of a triangle are 3, 5, 7. Find the measure of all the angles.
3. The sides of a triangle are 2, 3, 6. Find the largest angle.
4. The sides of a triangle are 8, 15, 20. Is the triangle acute, right or obtuse?
5. A triangular plot of land is founded by three streets which intersect in pairs at three places. The lengths of the sides of the triangle are 120 meters, 160 meters, and 170 meters. Find, to the nearest degree the measure of the largest angle formed.

SOLUTIONS TO APPLICATION PROBLEMS:

1. The smallest angle would be opposite the smallest side.

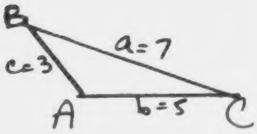
$$\text{Thus } 9^2 = 12^2 + 13^2 -2(12)(13)\cos \measuredangle A$$

$$81 = 144 + 169 - 312\cos \measuredangle A$$

$$\cos \measuredangle A = \frac{232}{312} = 0.7436$$

$$m \measuredangle A \approx 42.$$

2.



First, finding m $\angle$ A

$7^2 = 3^2 + 5^2 - 2(3)(5)\cos \angle A$

$49 = 9 + 25 - 30\cos \angle A$

$\cos \angle = \frac{-15}{30} = -0.5000$

m $\angle$ A = 120

m $\angle$ B or m $\angle$ C may be found using either the Law of Cosines or the Law of Sines:

$$\frac{\sin \angle B}{5} = \frac{\sin 120°}{7} = \frac{0.8600}{7}$$

$\sin \angle B = 0.6186$, m $\angle$ B $\simeq$ 38

m $\angle$ C can then be found by using

m $\angle$ C = 180 - (m $\angle$ A + m $\angle$ B).

m $\angle$ C = 22

3. This is impossible since the use of the lengths of two sides of a triangle must exceed the length of the third side. If the Law of Cosines is employed, the student should obtain $\cos \angle A > 1$ which has no solution.

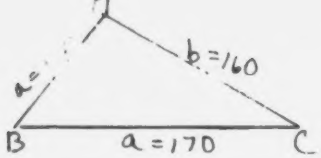
4. $20^2 = 8^2 + 15^2 - 2(8)(15)\cos \angle A$

   $400 = 64 + 225 - 240\cos \angle A$

   $\frac{-111}{240} = \cos \angle A$

   Since $\cos \angle A$ is negative, $\angle$ A (and the Δ) must be obtuse or since $20^2 > 8^2 + 15^2$, Δ is obtuse

5.



$17^2 = 16^2 + 12^2 - 2(16)(12)\cos \angle A$

$289 = 256 + 144 -384\cos \angle A$

$\cos \angle A = \frac{111}{384} = \frac{37}{128} = 0.2891$

m $\angle$ A = 73 (to nearest degree)

SUMMARY:

Ask, "If in triangle ABC, $c \geq a$, $c \geq b$, then what type of angle is $\measuredangle C$ when

1) $c^2 > a^2 + b^2$? (obtuse)
2) $c^2 < a^2 + b^2$? (acute)
3) $c^2 = a^2 + b^2$? (right)

LESSON: 31

AIM: How do we determine which method to use in solving triangles?

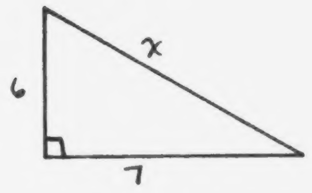
PERFORMANCE OBJECTIVES: The student will be able to...

1. Determine if simple right triangle trigonometry (including Pythagorean theorem) is sufficient to solve a triangle.
2. Apply the Law of Sines when AAS, SSA or ASA, is known.
3. Apply the Law of Cosines when SAS or SSS is known.

MOTIVATION:

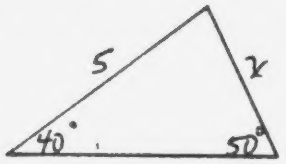
Write an equation which could be used to solve for the indicated side or angle?

a)



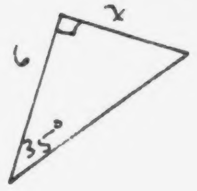
$$x^2 = 6^2 + 7^2$$

b)



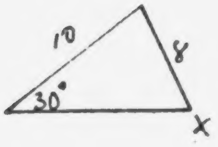
$$\frac{x}{\sin 40^\circ} = \frac{5}{\sin 50^\circ}$$

c)



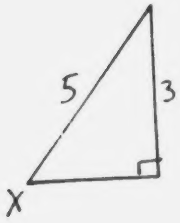
$$\tan 35^\circ = \frac{x}{6}$$

d)



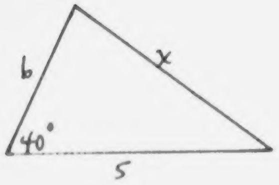
$$\frac{\sin \angle x}{10} = \frac{\sin 30^\circ}{8}$$

e)



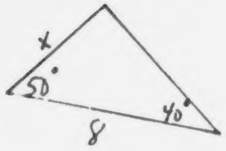
$\sin \measuredangle x = \frac{3}{5}$

f)



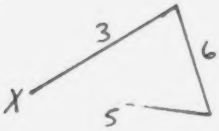
$x^2 = 6^2 + 5^2 - 2(6)(5)\cos 40^\circ$

g)



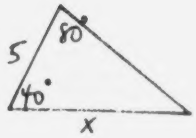
$\cos 50^\circ = \frac{x}{8}$

h)



$6^2 = 5^2 + 3^2 - 2(5)(3)\cos \measuredangle x$

i)



$$\frac{x}{\sin 80^\circ} = \frac{5}{\sin 60^\circ}$$

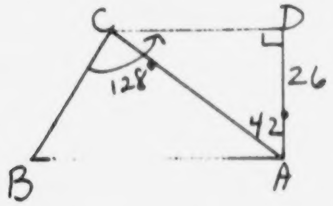
DEVELOPMENT:

In the above, a, c, e, g are illustrations of right triangle trigonometry; a simple application of sin, cos, tan or the Pythagorean theorem are sufficient to find unknown sides or angles.

Exercises b, d, i, are applications of the Law of Sines since the parts known are in the forms SAA, SSA or ASA.

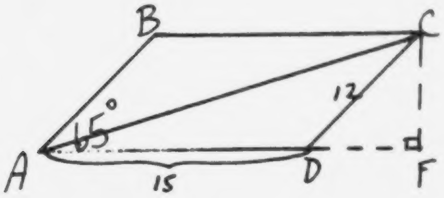
Exercises f, h, are applications of the Law of Cosines since the parts known are in the forms SAS or SSS.

APPLICATIONS:



1. In trapezoid ABCD, $\overline{AB}$ and $\overline{DC}$ are parallel bases and $\overline{AD} \perp \overline{DC}$. m $\measuredangle$ BCD = 128, m $\measuredangle$ DAC = 42 and AD = 26. Find AB to the nearest integer.

2. Two sides and the included angle of a parallelogram ABCD measure 12, 15 and 65° respectively, as shown. Find the length of the longer diagonal, and altitude $\overline{CF}$.



3. Find the perimeter of a regular pentagon inscribed in a circle of radius 8 cm.

4. The diagonals of a ▱P are 60 and 120 and one side is 75. Find the measure of the angle between the diagonals.

5. Two sides of a ▱P are 24 cm and 10cm. If the area is 120 cm^2, find an obtuse $\angle$ of the ▱P . Find the length of the longer diagonal.

SOLUTIONS TO APPLICATION PROBLEMS:

1. $\cos 42^\circ = \frac{26}{AC} = 0.7431$ or $AC = 35.0$

   Since $m \angle ACD = 48$ and $m \angle CBA = 52$, we determine $m \angle BCA = 80$.

   Then $\frac{AB}{\sin 80^\circ} = \frac{35.0}{\sin 52^\circ}$ or $AB = \frac{35\ (0.9848)}{0.7880} = 44$ (to nearest integer)

2. $AC^2 = 12^2 + 15^2 - 2(12)(15)\cos 115^\circ$

   $= 144 + 225 - 360(-0.4226) = 521.1$

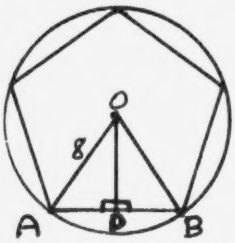
   $AC \approx 22.8$

   $\sin \angle CDF = \frac{CF}{12}$

   $\sin 65^\circ = \frac{CF}{12} = 0.9063$

   $CF \approx 10.9$

3.



$m \measuredangle AOB = \frac{1}{5}(360) = 72$

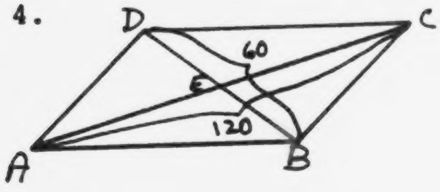
$\therefore m \measuredangle AOD = 36$

$\sin 36^\circ = \frac{AD}{8}$

$AD = 0.5878\ (8) = 4.7$ cm

$\therefore AB = 2(4.7) = 9.4$ cm and the perimeter $= 5(9.4) = 47.0$ cm

4.



Since AE = 60 and ED = 30

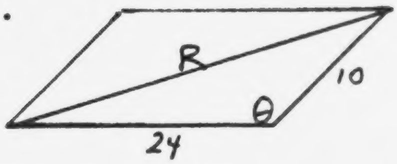
$75^2 = 60^2 + 30^2 - 2(60)(30)\cos \measuredangle AED$

$5625 = 3600 + 900 - 2400\cos \measuredangle AED$

$\cos \measuredangle AED = 0.4688$

$m \measuredangle AED \approx 62$

5.



$A = 24(10)\sin \theta = 120$

$\sin \measuredangle \theta = \frac{120}{240} = 0.5000$

$m \measuredangle \theta = 150$

$$
\begin{aligned}
R^2 &= 24^2 + 10^2 - 2(24)(10)\cos 150^\circ \\
&= 576 + 100 - 480(-0.8660) \\
&= 676 + 415.7 = 1091.7 \\
R &= \sqrt{1091.7} \quad 33 \text{ cm.}
\end{aligned}
$$

SUMMARY:

1. What do we use to solve for missing parts of right triangles? (sin, cos, tan, Pythagorean theorem)
2. When do we use the Law of Sines? (When AAS, ASA, or SSA is known.)
3. When do we use the Law of Cosines? (When SAS or SSS is known.)

LESSON: 32

AIM: How can forces with magnitude and direction be represented?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define and apply the concept of a vector force.
2. Use vectors to solve parallelogram of forces problems.

VOCABULARY: vector, resultant, scalar

MOTIVATION:

Present the following situation:

Two boys kick a soccer ball simultaneously. Can we determine the path the ball will follow? Take a specific example. Suppose the first boy's kick by itself would have sent the ball 60 ft. to the northeast, and that the second boy's kick would have sent it 50 feet to the East. In what direction will it travel? Where will the ball land?

DEVELOPMENT:

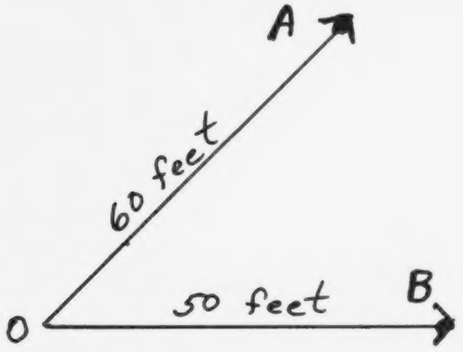
Be sure to clarify the physical principles involved in the motivation problem. Point out that each kick is an example of a force, which in turn is an example of a vector, since the force has both magnitude and direction. Give other examples of forces, e.g. pulling a sled, the force of gravitation (attractive force of the earth on an object.)

Explain that a quantity that has magnitude only is called a scalar. Have the class distinguish between velocity and speed.

Velocity is a vector, since it has a magnitude and direction, whereas speed is a scalar since it has only magnitude. Thus, we say that the speed of the wind is 10 m.p.h. but the velocity of the wind is 10 m.p.h. from the north.

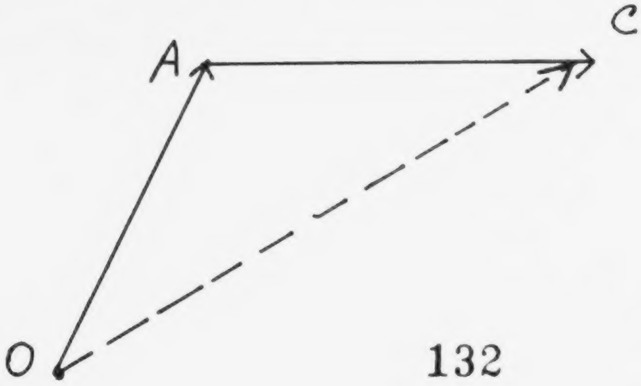
Students should also observe that a directed distance is a vector but a distance is a scalar.

In order to answer the challenge problem, we make use of a physical principle based upon Newton's Second Law of Motion. The principle states that if two or more forces act upon an object, each force produces its own effect, irrespective of the actions of the other forces. We can now represent the effect of each kick by a vector, using any convenient unit to represent distance, e.g. 1" = 10 ft.

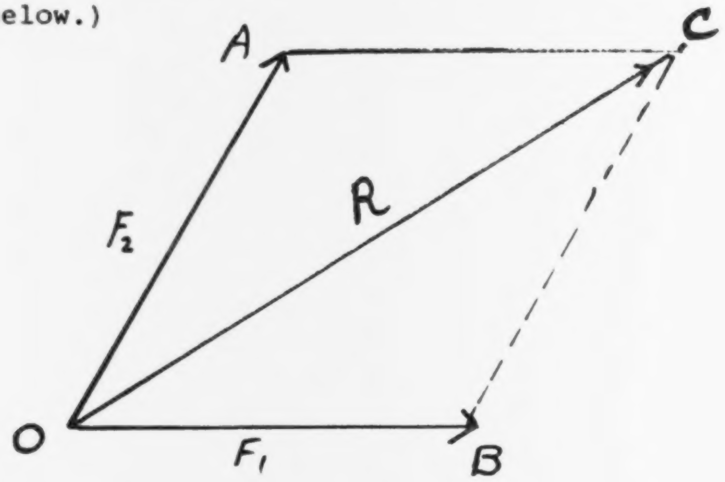


Students should be able to explain why the angle between the two forces in the challenge problem (AOB) = 45°. The magnitude of the vector is represented by the length of the line, the direction by the angle the line makes with any given line.

Elicit that the ball will move along a line inside angle AOB. Can we determine its path more accurately? Using the physical principle mentioned previously, we assume that the first boy alone kicks the ball, which then moves from O to A. The second boy then kicks the ball and it moves a distance equal to $\overline{OB}$ and parallel to $\overline{OB}$, and will end at C. Thus the ball started at O and ended up at C. If both boys had kicked the ball simultaneously, its path would be be $\overline{OC}$.



To summarize, the motion from 0 to B "added" to the motion from O to A produces a resulting motion from O to C. (See diagram below.)

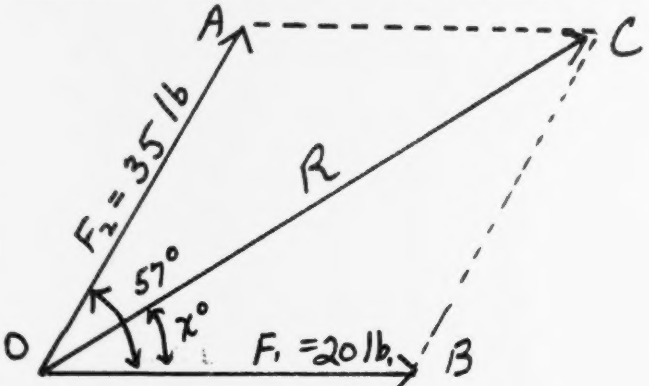


By drawing $\overline{OB}$ and $\overline{BC}$, the students should be able to prove that the figure OBCA is a parallelogram ($\overline{AC} = \overline{OB}$ and $\overline{AC} \parallel \overline{OB}$).

Have students generalize the situation. Elicit: **If two forces act on an object simultaneously, the resulting force is the diagonal of the parallelogram two of whose adjacent sides represent the forces.**

Tell the class that the parallelogram is called the "**parallelogram of forces**", $\overline{OC}$ is call the "**resultant**" of the two forces, and $\overline{OA}$ and $\overline{OB}$ are called "components" of $\overline{OC}$.

Now give the class an illustrative problem. Two forces of 20 lbs. and 35 lbs. act on an object. The angle between the forces is 57°. Find the magnitude of the resultant and the angle that the resultant makes with the smaller force.



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The student should suggest using the Law of Cosines to find R. $R^2 = (20)^2 + (35)^2 - 2(20)(35)\text{Cos } 123^\circ$

$\therefore$ $R = 49$ lbs. (approx.)

They should also suggest using the Law of Sines to find angle x.

$\text{Sin } x = \frac{35 \sin 57^\circ}{49}$ $\quad\therefore m \measuredangle x = 37$ (approx.)

Point out that the resultant makes the larger angle with the smaller force. Discuss the physical significance of this fact.

Have the class investigate the effect on the magnitude of the resultant if the angle between the two given forces changes. Questions, such as the following, should be considered:

1) When is the magnitude of the resultant force equal to the sum of the magnitudes of the two given forces?

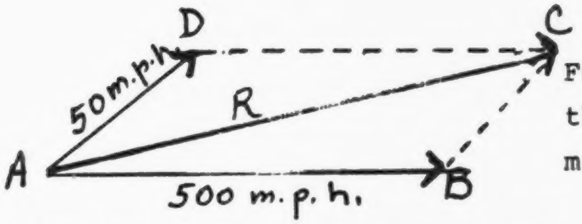
2) When is the magnitude of the resultant force equal to the difference in magnitudes of the two given forces?

3) Is it possible for the magnitude of the resultant force to be less than the magnitude of the smaller force?

4) When is the magnitude of the resultant force equal to 0?

APPLICATIONS:

1. Use vector addition of an airplane's velocity in still air and the wind velocity to determine the ground speed R.



For instance, find R to the nearest integer if $m\measuredangle DAB = 80$

2. Find to the nearest degree the measure of the angle between two forces of 30 lb. and 35 lb. if the magnitude of the resultant is 42 lb.

3. Two forces act on a point at an angle of 102°. The first is a force of 130 pounds. If the resultant makes an angle of $38^{\circ}\ 20'$ with the first force, what is the magnitude of the resultant to the nearest pound?

4. Two forces of 85 lbs. and 123 lbs. act at an angle of $128^{\circ}\ 40'$. Find to the nearest tenth the magnitude of their resultant.

5. Two forces of 10 lbs. and 20 lbs. act on angle of 90°. Find the magnitude and direction of the resultant.

SOLUTIONS TO APPLICATION PROBLEMS:

1. In $\triangle CAB$, $m \measuredangle B = 100$ and $CB = 50$.

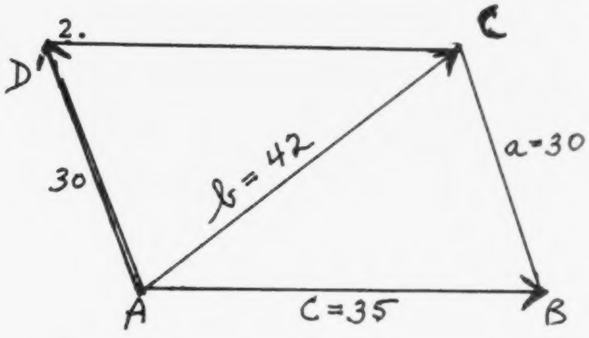
$$R^2 = 500^2 + 50^2 - 2(500)(50)\cos 100^{\circ}$$

$$= 250{,}000 + 2{,}500 - 50000(-0.1736) = 261{,}180$$

$R \approx 511$ mph

NOTE: Arithmetic may be simplified by dividing the sides by a constant (for instance 10), computing the new resultant and then multiplying back by the constant.

2.



$$42^2 = 30^2 + 35^2 - 2(30)(35)\cos \measuredangle B$$

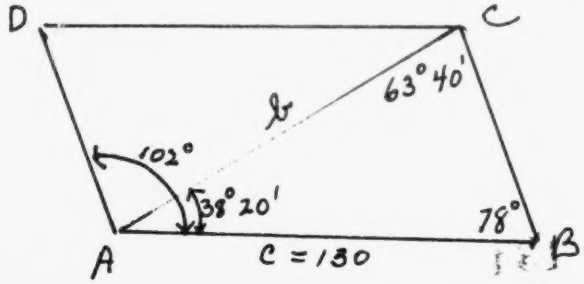
$$1764 = 900 + 1225 - 2100\cos \measuredangle B$$

$$\cos \angle B = \frac{361}{2100} = 0.1719$$

$m \measuredangle B = 80$ (to nearest degree)

$\therefore\ m \measuredangle DAB \approx 100^{\circ}$

3.

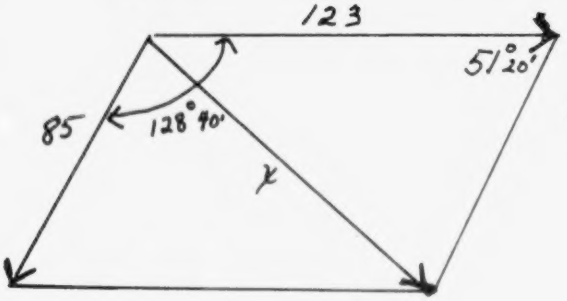


$$\frac{b}{\sin 78^{\circ}} = \frac{130}{\sin 63^{\circ}\ 40'}$$

$$b = \frac{(130)(0.9781)}{0.8962}$$

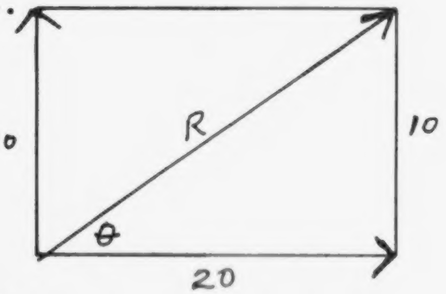
= 142 pounds (to nearest pound)

4.



$$\begin{aligned} x^2 &= 123^2 + 85^2 - 2(123)(85)\cos 51^\circ\ 20' \\ &= 15129 + 7225 - 20910(0.7808) \\ &= 22354 - 16327 = 6027 \\ x &= 6027 = 77.6 \text{ lbs.} \end{aligned}$$

5.



$\tan \angle \theta = \frac{10}{20} = 0.5000$

$m \angle \theta \quad 26^\circ\ 30'$

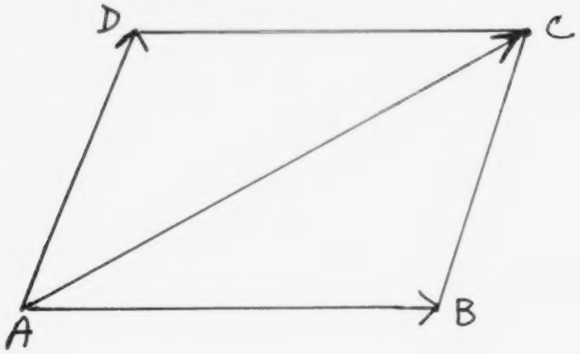
$\sin 26^\circ\ 30' = \frac{10}{R}$

$0.4462 = \frac{10}{R} \qquad R = 22.4$ lbs.

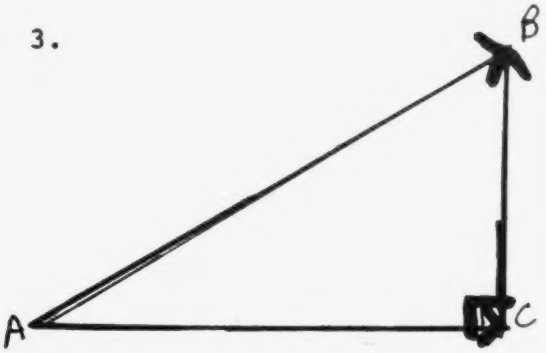
SUMMARY:

1. Ask "What is the difference between a vector and a scalar?"

2. "In the parallogram of forces shown, identify the components of $\overline{AC}$ and the resultant."



3.



Show why $\overline{BC} = \overline{AB} \sin \angle A$,
$\overline{AC} = \overline{AB} \cos \angle A$.

LESSON: 33

AIM: How can we use the sum and difference formulas?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Find sin (x+y), cos (x+y), sin (x-y), cos (x-y) given a well-chosen pair of the following: sin x, cos x, tan x, sin y, cos y, tan y and the location of the terminal sides of angles x and y.
2. Use the sum and difference formulas to prove some of the reduction formulas.
3. Evaluate an expression using one of these formulas.

VOCABULARY: identity

MOTIVATION:

Do Now: True or False?

1. $\cos(90^\circ - 45^\circ) = \cos 90^\circ - \cos 45^\circ$
2. $\sin(30^\circ + 60^\circ) = \sin 30^\circ + \sin 60^\circ$

1. $\cos(90^\circ - 45^\circ) = \cos 45^\circ = \frac{\sqrt{2}}{2}$

   $\cos 90^\circ - \cos 45^\circ = 0 - \frac{\sqrt{2}}{2} \neq \frac{\sqrt{2}}{2} \neq \frac{\sqrt{2}}{2}$

   $\therefore \cos(90^\circ - 45^\circ) = \cos 90^\circ - \cos 45^\circ$ is FALSE

2. $\sin(30^\circ + 60^\circ) = \sin 90^\circ = 1$

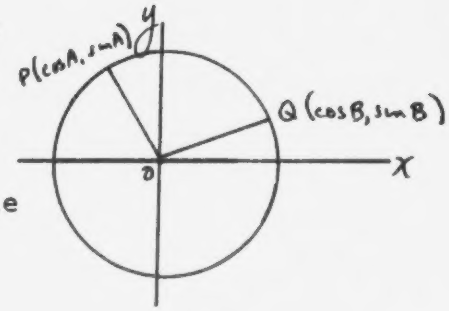
   $\sin 30^\circ + \sin 60^\circ = \frac{1}{2} + \frac{\sqrt{3}}{2} = \frac{1+\sqrt{3}}{2} \neq 1$

   $\therefore \sin(30^\circ + 60^\circ) = \sin 30^\circ + \sin 60^\circ$ is FALSE

Could cos(A-B) = cos A - cos B be an identity?
Why not?
Could sin(A+B) = sin A + sin B be an identity?
Why not?

DEVELOPMENT:

1. Derivation of a formula for cos (A-B)



Consider a unit circle with its center at the origin. Let the intersection of the unit circle with the positive portion of the x-axis be point X. Place points Q and P on the unit circle such that m ∡ QOX = B and m∡ POX = A. What is m ∡ POQ? What are the coordinates of P and Q? Find $(PQ)^2$ using the distance formula:

$$(PQ)^2 = (\cos A - \cos B)^2 + (\sin A - \sin B)^2$$

$$= (\cos^2 A - 2\cos A \cos B + \cos^2 B) + (\sin^2 A - 2\sin A \sin B + \text{Sin}^2 B)$$

$$= (\cos^2 A + \sin^2 A) + (\cos^2 B + \sin^2 B) - 2(\cos A \cos B - 2\sin A \sin B + \sin A \sin B)$$

$$= 2 - 2(\cos A \cos B + \sin A \sin B.)$$

We may rotate ∡ POQ about point 0 throuth an angle of B° in a clockwise direction. What are the images of points P and Q? The coordinates of Q' are (1, 0) and P' are (cos (A -B), sin (A - B)). Again, using the distance formula:

$$(P'Q')^2 = (\cos(A - B) - 1)^2 + \left[\sin(A - B)\right]^2$$

$$= \cos^2(A - B) - 2\cos(A - B) + 1 + \sin^2(A - B)$$

$$= 2 - 2\cos(A - B)$$

Since PQ = P'Q',

$$2 - 2(\cos A \cos B + \sin A \sin B)$$

$$= 2 - 2\cos(A - B)$$

$$-2(\cos A \cos B + \sin A \sin B) = -2\cos(A - B)$$

Therefore, **cos(A - B) = cos A cos B + sin A sin B.**

Notice that this derivation places no restrictions on the angles, A and B.

2. Do Applications 1-4.

3. Then: Show cos A = sin(90 - A) using application problem no. 3.

   letting B = 90 - A.

   Show sin (-A) = - sin A again using application. problem no. 3

   and letting B = - A.

4. How can we use cos(A-B) formula to find a formula for cos(A+B)? Elicit we need to express A+B as a difference.

   cos(A + B) = cos (A - (- B))

   Elicit: = cos A cos(- B) + sin A sin(- B)

   = cos A cos B + sin A(- sin B)

   = **cos A cos B - sin A sin B.**

5. How can we use our previous formula to find sin(A+B)?

   Elicit:

   sin(A + B) = cos (90 - (A + B)) = cos [(90 - A) - B]

   = cos (90 - A) cos B + sin (90 - A) sin B

   = **sin A cos B + cos A sin B.**

6. How can we find sin(A-B) using our previous formulas?

   Elicit:

   sin(A - B) = sin (A + (- B)) = sin A cos (- B) + cos A sin (-B)

   = sin A cos B + cos A(- sin B)

   = **sin A cos B - cos A sin B.**

7. Do Applications 5 and 6.

APPLICATIONS:

1. Find cos 15° using the new formula

2. Solve for x: cos x° = cos 70°cos 63° + sin 70°sin 63°

3. Show cos (90° - B) = sin B

4. Show cos (- A) = cos A

(For 5 and 6.) If sin A = $\frac{5}{13}$, sin B = $\frac{3}{5}$, ∡ A is in guadrant II, and ∡ B is quadrant I

5. Find cos (A + B)

6. Find sin (A + B)

SOLUTIONS TO APPLICATION PROBLEMS:

1. $\cos 15^\circ = \cos(45^\circ - 30^\circ)$

   $\cos(A - B) = \cos A \cos B + \sin A \sin B$

   $\cos(45^\circ - 30^\circ) = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$

   $$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$$

   $$\cos 15^\circ = \frac{\sqrt{6} + \sqrt{2}}{4}$$

2. Since $\cos A \cos B + \sin A \sin B = \cos(A - B)$,

   $\cos 70^\circ \cos 63^\circ + \sin 70^\circ \sin 63^\circ = \cos(70^\circ - 63^\circ) = \cos 7^\circ$

   $\therefore \quad x = 7^\circ$

3. $\cos(A - B) = \cos A \cos B + \sin A \sin B$

   $\cos(90^\circ - B) = \cos 90^\circ \cos B + \sin 90^\circ \sin B$

   $= 0 \cdot \cos B + 1 \cdot \sin B$

   $= \sin B$

4. $\cos(A - B) = \cos A \cos B + \sin A \sin B$

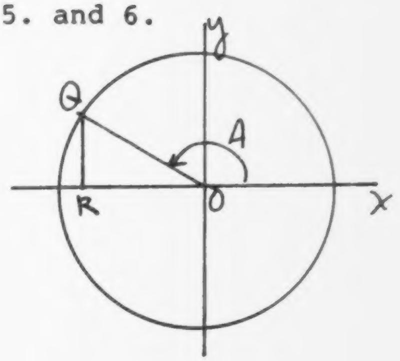
   $\cos(-A) = \cos(0^\circ - A)$

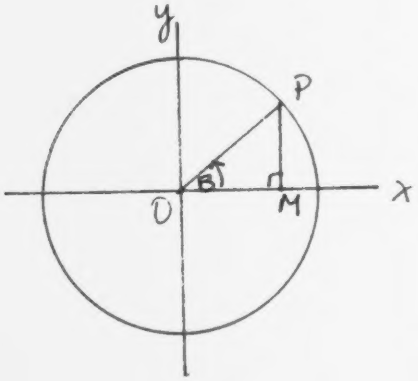
   $= \cos 0^\circ \cos A + \sin 0^\circ \sin A$

   $= 1 \cdot \cos A + 0 \cdot \sin A$

   $= \cos A.$

5. and 6.





If QO = 13, what is length of QR? 5

OR? 12.

What is cos A? $\frac{-12}{13}$

If OP = 5 and PM = 3, what is length of OM? 4

What is cos B? $\frac{4}{5}$

$$\begin{aligned}\cos(A + B) &= \cos A \cos B - \sin A \sin B \\ &= \frac{-12}{13} \cdot \frac{4}{5} - \frac{5}{13} \cdot \frac{3}{5} = \frac{-63}{65}.\end{aligned}$$

$$\begin{aligned}\sin(A + B) &= \sin A \cos B + \cos A \sin B \\ &= \frac{5}{13} \cdot \frac{4}{5} + \frac{-12}{13} \cdot \frac{3}{5} = \frac{-16}{65}.\end{aligned}$$

SUMMARY:

What are the similarities in the formulas for cos $(A \pm B)$ and sin $(A \pm B)$? differences?

LESSON: 34

AIM: What are the double angle formulas?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Derive the formulas for sin 2x and cos 2x
2. Use the double angle formulas to find sin 2x or cos 2x given the value of sin x, cos x
3. Evaluate an expression applying these formulas.

MOTIVATION:

True or False?
1. $\sin 60^{\circ} = 2\sin 30^{\circ}$
2. $\cos 90^{\circ} = 2\cos 45^{\circ}$

1. $\sin 60^{\circ} = \frac{\sqrt{3}}{2}$; $2\sin 30^{\circ} = 2 \cdot \frac{1}{2} = 1 \neq \frac{\sqrt{3}}{2}$

   $\therefore \sin 60^{\circ} = 2\sin 30^{\circ}$ is FALSE!

2. $\cos 90^{\circ} = 0 \quad 2\cos 45^{\circ} = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2} \neq 0$

   $\therefore \cos 90^{\circ} = 2\cos 45^{\circ}$ is FALSE!

Is sin 2x = 2sin x?...Why not?
Is cos 2x = 2cos x?...Why not?

DEVELOPMENT:

In order to derive a formula for sin 2A, which of our trigonometric formulas do we need? Elicit: sin (A + B)
How can we express 2A as a sum? Elicit: 2A = A + A
What substitution will help us in the derivation?
Let B = A.

$$\begin{aligned} \sin (A + B) &= \sin A \cos B + \cos A \sin B \\ \sin 2A = \sin (A + A) &= \sin A \cos A + \cos A \sin A \\ &= \sin A \cos A + \sin A \cos A \end{aligned}$$

$$\underline{\sin 2A = 2 \sin A \cos A}$$

How can we do a similar derivation for cos 2A?

$$\cos (A + B) = \cos A \cos B - \sin A \sin B$$

$$\cos 2A = \cos (A + A) = \cos A \cos A - \sin A \sin A$$

$$\underline{\cos 2A = \cos^2 A - \sin^2 A}$$

Which identity can we use to express cos 2A in terms of cos A only? sin A only?

$$\sin^2 A + \cos^2 A = 1$$

$$\therefore \sin^2 A = 1 - \cos^2 A$$

$$\therefore \cos 2A = \cos^2 A - (1 - \cos^2 A)$$

$$\therefore \underline{\cos 2A = 2\cos^2 A - 1}$$

or

$$\cos^2 A = 1 - \sin^2 A$$

$$\cos 2A = (1 - \sin^2 A) - \sin^2 A$$

$$\underline{\cos 2A = 1 - 2 \sin^2 A}$$

APPLICATIONS:

1. Evaluate $2\sin 15° \cos 15°$

2. Evaluate $\cos^2 75° - \sin^2 75°$

3 If $\cos x = \frac{8}{17}$, find cos 2x

(For 4. and 5.) If $\tan x = \frac{3}{4}$ and x is in quadrant III, find

4. sin 2x

5. cos 2x

**6. Derive a formula for sin 3x only in terms of sin x.

SOLUTIONS TO APPLICATION PROBLEMS:

1. $2\sin x \cos x = \sin 2x$

$$\therefore \quad 2\sin 15° \cos 15° = \sin 2 \cdot 15° = \sin 30° = \frac{1}{2}$$

2. $\cos^2 x - \sin^2 x = \cos 2x$

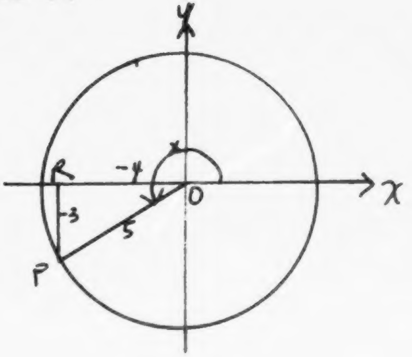
$$\therefore \quad \cos^2 75° - \sin^2 75° = \cos 2 \cdot 75° = \cos 150°$$

$$= - \cos 30° = \frac{-\sqrt{3}}{2}.$$

3. $\cos 2x = 2\cos^2 x - 1$

$$= 2 \cdot \left(\frac{8}{17}\right)^2 - 1 = \frac{128}{289} - 1 = \frac{-161}{289}$$

4. and 5.



$\sin x = \frac{-3}{5}$

$\cos x = \frac{-4}{5}$

4. $\sin 2x = 2\sin x \cos x$

$= 2 \cdot \frac{-3}{5} \cdot \frac{-4}{5} = \frac{24}{25}$

5. $\cos 2x = \cos^2 x - \sin^2 x$

$= (\frac{-4}{5})^2 - (\frac{-3}{5})^2$

$= \frac{16}{25} - \frac{9}{25} = \frac{7}{25}$

*6. How can we express 3x in a form that will enable us to use our recently derived formulas?

$\sin 3x = \sin (2x + x) = \sin 2x \cdot \cos x + \cos 2x \cdot \sin x$

$= 2\sin x \cdot \cos x \cdot \cos x + (1 - 2 \sin^2 x)\sin x$

$= 2\sin x \cdot \cos^2 x + \sin x - 2\sin^3 x$

$= 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x$

$= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x$

$\therefore \sin 3x = 3\sin x - 4\sin^3 x.$

SUMMARY:

How are today's new formulas related to yesterday's formulas?

LESSON: 35

AIM: What are the half-angle formulas?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Derive the formula for $\sin \frac{x}{2}$ and $\cos \frac{x}{2}$ from the formula for cos 2x.
2. Find the $\sin \frac{x}{2}$ or $\cos \frac{x}{2}$, given the location of the terminal side of angle x and the value of sin x, cos x or tan x.
3. Evaluate a specific trigonometric function of a particular angle using the appropriate half-angle formula.

MOTIVATION:

1. Is $\sin 45^\circ = \frac{1}{2}\sin 90^\circ$
2. Is $\cos 30^\circ = \frac{1}{2}\cos 60^\circ$?

1. $\sin 45^\circ = \frac{\sqrt{2}}{2}$
   $\frac{1}{2}\sin 90^\circ = \frac{1}{2} \cdot 1 = \frac{1}{2} \neq \frac{\sqrt{2}}{2}$. $\therefore$ No, $\sin 45^\circ \neq \frac{1}{2}\sin 90^\circ$

2. $\cos 30^\circ = \frac{\sqrt{3}}{2}$ .866 $\quad \frac{1}{2}\cos 60^\circ = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \neq \frac{\sqrt{3}}{2}$.
   No, $\cos 30^\circ \neq \frac{1}{2}\cos 60^\circ$.

What can we conclude about $\sin \frac{x}{2}$ as compared with sin x?

What can we conclude about $\cos \frac{x}{2}$ as compared with cos x?

DEVELOPMENT:

Let's derive formulas for $\sin \frac{x}{2}$ and $\cos \frac{x}{2}$.

We have formulas for functions of 2x in terms of functions of x.

How can we obtain formulas for functions of $\frac{A}{2}$ in terms of functions of A?

How can $\cos 2x = 1 - 2\sin^2 x$ be written using angles A and $\frac{A}{2}$?

How can we justify the replacement?

Elicit: Let $2x = A$, then $x = \frac{A}{2}$.

$\cos 2x = 1 - 2\sin^2 x$	$\cos 2x = 2\cos^2 x - 1$
$\cos A = 1 - 2\sin^2(\frac{A}{2})$	$\cos A = 2\cos^2(\frac{A}{2}) - 1$
$\cos A - 1 = -2\sin^2(\frac{A}{2})$	$1 + \cos A = 2\cos^2(\frac{A}{2})$
$\frac{1 - \cos A}{2} = \sin^2(\frac{A}{2})$	$\frac{1 + \cos A}{2} = 2\cos^2(\frac{A}{2})$
$\sin(\frac{A}{2}) = \pm\sqrt{\frac{1 - \cos A}{2}}$	$\cos(\frac{A}{2}) = \pm\sqrt{\frac{1 + \cos A}{2}}$

What is the significance of the "±" sign?

Under what conditions will we use the positive portion?

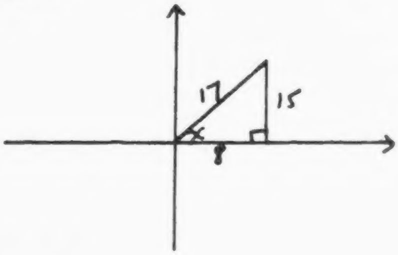
The negative portion?

APPLICATIONS:

1. If $\sin x = 15/17$ and x is acute, find $\cos \frac{x}{2}$.
2. What effect will the following change in problem no. 1 have on the result: x is in quadrant IV?
3. Evaluate $\cos 75^\circ$ using the appropriate half-angle formula.
4. Evaluate $\sin(-15^\circ)$ using the appropriate half-angle formula.

SOLUTIONS TO APPLICATION PROBLEMS:

1. If $0^\circ < x < 90^\circ$, then $0^\circ < \frac{x}{2} < 45^0$. $\therefore$ we take positive square root.



$\sin x = 15/17$

$\therefore \cos x = 8/17$

$$\cos \frac{X}{2} = \sqrt{\frac{1 + 8/17}{2}} = \sqrt{\frac{25}{34}} = \frac{5}{\sqrt{34}}$$

2. If x is in quadrant IV, then $270^\circ < x < 360^\circ$;
   ∴ $135^\circ < \frac{x}{2} < 180^\circ$.

   ∴ $\frac{x}{2}$ is in quadrant II. Since cosine of a quadrant II angle is negative, we say $\cos\frac{x}{2} = -\sqrt{\frac{1 + \cos x}{2}} = \frac{-5}{\sqrt{34}}$.

3. $\cos 75^\circ = \cos\frac{150^\circ}{2}$. So $x = 150^\circ$

   75° is in quadrant I ∴ $\cos\frac{x}{2} = \sqrt{\frac{1 + \cos x}{2}}$

   $\cos 75^\circ = \sqrt{\frac{1 + \cos 150}{2}} = \sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2 - \sqrt{3}}{4}}$

4. $\sin(-15^\circ) = \sin\frac{-30^\circ}{2}$. So $x = -30^\circ$

   -15° is in quadrant IV ∴ $\sin\frac{x}{2} = -\sqrt{\frac{1 - \cos x}{2}}$

   $\sin(-15^\circ) = -\sqrt{\frac{1 - \sqrt{3}/2}{2}} = -\sqrt{\frac{2 - \sqrt{3}}{4}}$

SUMMARY:

What are the half-angle formulas?

How do today's formulas relate to our previously learned formulas?

LESSON: 36

AIM: What formulas do we use for the tangent function?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Find tan (x + y) or tan (x - y) using the appropriate formula, given a well-chosen part of the following: sin x, cos x, tan x, sin y, cos y, tan y and the location of the terminal sides of angles x and y.

2. Find tan 2x or tan $\frac{x}{2}$, given sin x, cos x, or tan x and the location of the terminal side of angle x.

3. Derive the formula: $\tan \frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{1 + \cos x}}$

MOTIVATION:

Is tan (45° + 45°) = tan 45° + tan 45°?

tan (45° + 45°) = tan 90 which is undefined

tan 45° + tan 45° = 1 + 1 = 2 which is clearly not undefined.

$\therefore$ No, tan (45° + 45°) $\neq$ tan 45° + tan 45°.

DEVELOPMENT:

What identities can we use to derive a formula for tan (A + B)?

$\tan\theta = \frac{\sin\theta}{\cos\theta}$; $\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

How can these identities be related so that we can derive a formula for tan (A + B)?

$$\tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)} = \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$$

$$= \frac{\frac{\sin A\cos B}{\cos A\cos B} + \frac{\cos A\sin B}{\cos A\cos B}}{\frac{\cos A\cos B}{\cos A\cos B} - \frac{\sin A\sin B}{\cos A\cos B}}$$

* The teacher could suggest division of numerator and denominator of fraction by cos A cos B and can elicit that division distributes over addition and subtraction.

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

What identities can we use to derive a formula for tan(A -B)?

[A derivation is not required.]

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

What similarities exist in the 2 new formulas? differences?

How can we derive a formula for tan 2A?

Use $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$ and replace B with A

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

What identities can we use to derive a formula for $\tan \frac{x}{2}$?

$$\tan \frac{x}{2} = \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} = \frac{\pm\sqrt{\frac{1 - \cos x}{2}}}{\pm\sqrt{\frac{1 - \cos x}{2}}} = \pm\sqrt{\frac{1 - \cos x}{1 + \cos x}}$$

APPLICATIONS:

(Should be properly placed throughout the lesson.)

1. $\cos A = \frac{3}{5}$ and $\sin B = \frac{-12}{13}$. A is in quadrant I. B is in quadrant III. Find tan(A + B).
2. Use the formula for tan(x + y) to find tan 75°.
3. Express tan(225° - x) in terms of tan x.
4. Solve for θ: $\frac{\tan 140° - \tan 5°}{1 + \tan 140° \tan 5°} = \tan \theta$.

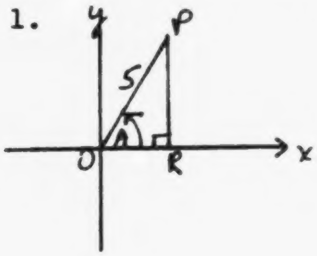
5. A and B are positive acute angles.
   $\cos A = \frac{1}{\sqrt{5}}$ and $\sin B = \frac{2}{\sqrt{13}}$
   Find $\tan(A - B)$.

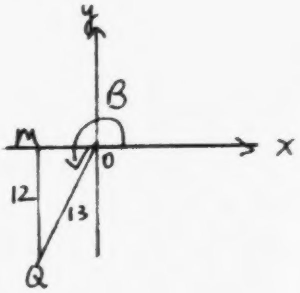
6. If $\cos x = \frac{4}{5}$ and x is in quadrant I, find $\tan \frac{x}{2}$.

7. If $\sin A = \frac{5}{13}$ and A is acute, find tan 2A.

SOLUTIONS TO APPLICATION PROBLEMS:

1.





We deduce that PR = 4 and OM = 5

$\therefore \tan A = 4/3$

and $\tan B = 12/5$

$$\therefore \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\frac{4}{3} + \frac{12}{5}}{1 - \frac{4}{3} \cdot \frac{12}{5}} = \frac{56}{-33} \text{ or } \frac{-56}{33}$$

2. What values of x and y are feasible for us to use? 45° 30°.
   Why are these values reasonable?

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°} = \frac{1 + \frac{1}{\sqrt{3}}}{1 - 1 \cdot \frac{1}{3}}$$

$$= \frac{1 + \frac{\sqrt{3}}{3}}{1 - \frac{\sqrt{3}}{3}} = \frac{3 + \sqrt{3}}{3 - \sqrt{3}} = \frac{3 + \sqrt{3}}{3 - \sqrt{3}} \cdot \frac{3 + \sqrt{3}}{3 + \sqrt{3}} = \frac{9 + 6\sqrt{3} + 3}{9-3}$$

$$= \frac{12 + 6\sqrt{3}}{6} = 2 + \sqrt{3}$$

3. What formula is suggested by the problem?

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

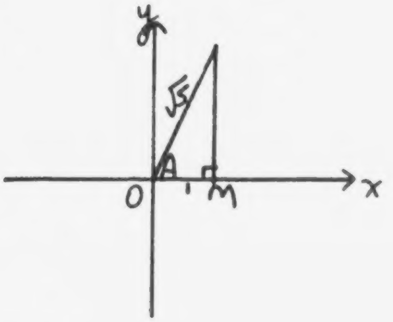
$$\tan(225° - x) = \frac{\tan 225° - \tan x}{1 + \tan 225° \tan x} = \frac{1 - \tan x}{1 + \tan x}$$

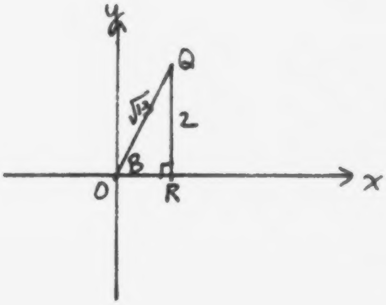
4. What formula does the left side of the equation recall for us?

$140° = A$ and $5° = B$ in $\tan (A - B)$.

$\therefore \theta = 140° - 5° = \underline{135}°$

5.





$$(PM)^2 + 1^2 = (5)^2$$
$$(PM)^2 = 4$$
$$PM = 2$$
$$\therefore \tan A = 2$$

$$(OR)^2 + 2^2 = (\sqrt{13})^2$$
$$(OR)^2 + 2 = 9$$
$$OR = 3$$
$$\therefore \tan B = \frac{2}{3}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} = \frac{2 - \frac{2}{3}}{1 + 2\cdot\frac{2}{3}} = \frac{4}{7}$$

6. What formula is necessary to solve this problem?

$$\tan \frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{1 + \cos x}}$$

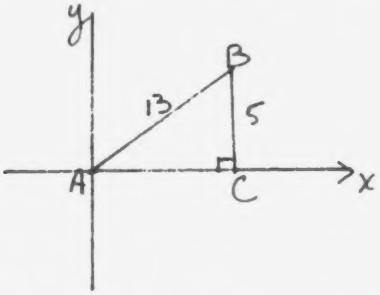
Since x is in the first quadrant, so is $\frac{x}{2}$

$$\ldots \tan \frac{x}{2} = \sqrt{\frac{1 - \frac{4}{5}}{1 + \frac{1}{5}}} = \sqrt{\frac{\frac{1}{5}}{\frac{6}{5}}} = \sqrt{\frac{1}{6}} = \frac{\sqrt{1}}{\sqrt{6}} = \frac{1}{\sqrt{6}} = \frac{\sqrt{6}}{6}$$

7. What formula is necessary to solve this problem?

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

How can we find tan A?



AC = 12

$\tan A = \frac{5}{12}$

$$\tan 2A = \frac{2 \cdot \frac{5}{12}}{1 - (\frac{5}{12})^2}$$

$$= \frac{\frac{5}{6}}{1 - \frac{25}{144}} = \frac{\frac{5}{6}}{\frac{119}{144}} = \frac{5}{6} \cdot \frac{144}{199}$$

$$= \frac{120}{119}$$

SUMMARY:

What are the formulas for tan(R+S), tan (R-S), tan 2R and tan $\frac{R}{2}$?

How can the formulas that we learned today be related to those learned earlier in this unit?

LESSON: 37

AIM: How do we apply the newer trigonometric formulas to proving identities?

PERFORMANCE OBJECTIVES: The student will be able to...

1. State the trigonometric formulas involving double angles, half angles, sums, or differences (as appropriate).
2. Prove trigonometric identities using formulas mentioned in objective no. 1.

MOTIVATION:

Prove the following identity for all values of x for which the expressions are defined.

$$\sin 2x = \frac{2\tan x}{1 + \tan^2 x}$$

What makes this identity different from any others that we have previously studied?

DEVELOPMENT:

Use the previous question to elicit that no previous identity has had in it a trigonometric function of a double angle.

What technique have we previously learned to prove identities? Change both sides to expressions using sines and cosines of single angles.

$$\sin 2x = \frac{2\tan x}{1 + \tan^2 x}$$

$$2\sin x \cos x \overset{?}{=} \frac{2\cdot\frac{\sin x}{\cos x}}{\sec^2 x}$$

Steps are
reversible

$$\frac{\frac{2\sin x}{\cos x}}{\frac{1}{\cos^2 x}}$$

$$\frac{2\sin x}{\cos x} \cdot \cos^2 x$$

$$2\sin x \cos x = 2\sin x \cos x$$

In the process of writing out the solution, a review of identities as needed should occur. Also, a review of algebraic ideas (such as division of fractions) will be necessary. Note that this solution is not unique but is only suggested.

APPLICATIONS:

Prove each of the following are identities for all values of x for which the expressions are defined.

1. $\tan x + \cot x = \frac{2}{\sin 2x}$

2. $\frac{\sin 2x}{\sin x} - \frac{\cos 2x}{\cos x} = \sec x$

3. $\tan\left(\frac{\pi}{4} + x\right) = \frac{\cos x + \sin x}{\cos x - \sin x}$

4. $\frac{2\sin A\cos A}{\cos^2 A - \sin^2 A} = \frac{2\tan A}{1 - \tan^2 A}$

SOLUTIONS TO APPLICATION PROBLEMS:

1. $\tan x + \cot x = \dfrac{2}{\sin 2x}$

$$\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} \overset{?}{=} \frac{2}{2\sin x \cos x}$$

$$\frac{\sin^2 x + \cos^2 x}{\sin x \cos x} \overset{?}{=} \frac{1}{\sin x \cos x}$$

$$\frac{1}{\sin x \cos x} = \frac{1}{\sin x \cos x}$$

2. $\dfrac{\sin 2x}{\sin x} - \dfrac{\cos 2x}{\cos x} = \sec x$

$$\frac{2 \sin x \cos x}{\sin x} - \frac{\cos 2x}{\cos x} \overset{?}{=} \frac{1}{\cos x}$$

$$2 \cos x - \frac{\cos 2x}{\cos x} =$$

$$\frac{2\cos^2 x - \cos 2x}{\cos x} =$$

$$\frac{2\cos^2 x - (2 \cos^2 x - 1)}{\cos x} =$$

$$\frac{1}{\cos x} = \frac{1}{\cos x}$$

3. $$\tan\left(\frac{\pi}{4} + x\right) = \frac{\cos x + \sin x}{\cos x - \sin x}$$

$$\frac{\tan\frac{\pi}{4} + \tan x}{1 - \tan\frac{\pi}{4}\tan x} =$$

$$\frac{1 + \tan x}{1 - \tan x} =$$

$$\frac{1 + \frac{\sin x}{\cos x}}{1 - \frac{\sin x}{\cos x}} =$$

$$\frac{\cos x + \sin x}{\cos x - \sin x} = \frac{\cos x + \sin x}{\cos x - \sin x}$$

4. $$\frac{2\sin A\cos A}{\cos^2 A - \sin^2 A} = \frac{2\tan A}{1 - \tan^2 A}$$

$$\frac{\sin 2A}{\cos 2A} = \tan 2A$$

$$\tan 2A = \tan 2A$$

SUMMARY:

What are the main techniques for proving identities? Responses should be noted on the chalkboard.

LESSON: 38

AIM: How do we apply the newer trigonometric formulas to solving trigonometric equations?

PERFORMANCE OBJECTIVES: The student will be able to...

1. State the relevant trigonometric formulas as appropriate.
2. Solve equations involving the double angle formulas to the nearest degree in the interval $0° \leq x < 360°$.

MOTIVATION:

Solve for x in the interval $0° \leq x < 360°$

1. $2\sin x - 1 = 0$
2. $\sin^2 x = \sin x$
3. $\sin 2x = \sin x$

Distinquish among the three equations on the board. Which types have we studied already? Trigonometric - linear and quadratic. Which type have we not yet studied? Trigonometric - involving newer formulas.

DEVELOPMENT:

Elicit the method of solution for each of the equations listed above.

1.
$2\sin x - 1 = 0$
$2\sin x = 1$
$\sin x = \frac{1}{2}$
$x = 30°$ or $150°$
$\{30°, 150°\}$

2.
$\sin^2 x = \sin x$
$\sin^2 x - \sin x = 0$
$\sin x (\sin x - 1) = 0$
$\sin x = 0$ or $\sin x - 1 = 0$
$\sin x = 1$
$x = 0^0$ or $180°$ or $x = 90°$
$\{0°, 90°, 180°\}$

How do we begin to solve most quadratic equations?

What error results from division by sin x in line 1?
(We lose 2 roots!)

3. $\sin 2x = \sin x$ What substitution can we make to enable us to have only trigonometric functions of a single angle?

$2\sin x \cos x = \sin x$ Should we divide by sin x? . . . Why not? What alternative do we have?

$2\sin x \cos x - \sin x = 0$

$\sin x(2\cos x - 1) = 0$

$\sin x = 0$ or $2\cos x - 1 = 0$

$\cos x = \frac{1}{2}$

$x = 0°$ or $180°$ or $x = 60°$ or $300°$

$\left\{0°, 60°, 180°, 300°\right\}$

APPLICATIONS:

Solve for x in the interval $0° \leq x < 360°$.
Give answers to the nearest minute if necessary.

1. $\cos 2x = \cos x$
2. $3\cos 2x + 8\sin x = -5.$

Which formula for cos 2x is more advantageous in each problem?

Why?

SOLUTIONS TO APPLICATION PROBLEMS:

1. $\cos 2x = \cos x$

$2\cos^2 x - 1 = \cos x$

$2\cos^2 x - \cos x - 1 = 0$

$(2\cos x + 1)(\cos x - 1) = 0$

$2\cos x + 1 = 0$ or $\cos x - 1 = 0$

$\cos x = \frac{-1}{2}$ or $\cos x = 1$

$x = 120°$ or $240°$ or $x = 0°$

$\left\{0°, 120°, 240°\right\}$

2. $3\cos 2x + 8\sin x = -5$

$$3(1 - 2\sin^2 x) + 8\sin x = -5$$

$$3 - 6\sin^2 x + 8\sin x = -5$$

$$0 = 6\sin^2 x - 8\sin x - 8$$

$$0 = 3\sin^2 x - 4\sin x - 4$$

$$0 = (3\sin x + 2)(\sin x - 2)$$

$$3\sin x + 2 = 0 \quad \text{or} \quad \sin x - 2 = 0$$

$$\sin x = \frac{-2}{3} \quad \text{or} \quad \sin x = 2$$

$$\sin x = -.6667 \quad \text{or} \quad \sin x = 2$$

$$x = 221°49' \quad \text{or} \quad 318°11'$$

$$\left\{221°49',\ 319°11'\right\}$$

Note that since $-1 \leq \sin x \leq 1$,

$\sin x = 2$ has no roots

$$10\left[\begin{array}{l} a\left[\begin{array}{r} \sin 41°40' = .6648 \\ \sin A = .6667 \end{array}\right]19 \\ \quad\ \sin 41°50' = .6670 \end{array}\right]22$$

$$\frac{a}{10} = \frac{19}{22}$$

$$22a = 190$$

$$a = 9$$

$$A = 41°49'$$

SUMMARY:

What previously learned ideas were useful in today's lesson?

1. Solving linear trigonometric equations.
2. Solving quadratic trigonometric equations.
3. Formulas for double angles.
4. Factoring trinomials.
5. Interpolation.

Cite specific applications of each from your notes.

LESSON: 39

AIM: How do we evaluate inverse trigonometric relations and functions?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Distinguish the meanings of the following notations:
   $\sin^{-1} x$, $\mathrm{Sin}^{-1} x$
   arc sin x, Arc sin x
2. Translate from inverse trigonometric notation to direct trigonometric notation and vice versa.
3. Evaluate expressions using principal value notation.

MOTIVATION:

Solve for x: $\sin x = \frac{1}{2}$

Students will respond with $\{30^\circ, 150^\circ\}$

Encourage other responses by noting that no restriction on the domain is given.

Students will respond with $\{30^\circ + k \cdot 360^\circ, 150^\circ + k \cdot 360^\circ\}$, where k is any integer.

(A graphical solution of $y = \sin x$ and $y = \frac{1}{2}$ will lead nicely into the development.)

DEVELOPMENT:

On the same set of axes, graph $y = \sin x$ and its inverse. Students will recall that the inverse of $y = \sin x$ is $x = \sin y$ from their earlier work on functions and that the graphs of inverses are reflections in the line whose equation is $y = x$.

Is $y = \sin x$ a function? Why?

Is $x = \sin y$ a function? Why not?

Where can we "cut" the graph of $x = \sin y$ in order that the resulting graph be a function?

Various answers are possible.

Introduce notation $y = \text{arc sin } x$ or $y = \sin^{-1} x$ for the relation $x = \sin y$ and $y = \text{Arc sin } x$ or $y = \text{Sin}^{-1} x$ for the function which is restricted by $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$

Read "$y = \text{Arc sin } x$ or $y = \text{Sin}^{-1} x$" as " is the principal value of the angle whose sine is x."

Other ranges of inverse trigonometric functions are:

for $y = \text{Arc cos } x$ $\quad o \leq y \leq \pi$

for $y = \text{Arc tan } x$ $\quad \frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$

APPLICATIONS:

Find the value of

1. $\text{Arc sin } \frac{1}{2}$
2. $\text{Arc cos } \frac{1}{2}$
3. $\text{Arc tan } (-1)$
4. $\text{Arc sin } 2$
5. $\text{Arc cos } 0$
6. $\sin(\text{Arc tan } 1)$
7. $\tan(\text{Arc cos } \frac{5}{13})$
8. $\cot(\text{Arc tan } \frac{7}{3})$
9. $\cos 2(\text{Arc sin } \frac{-3}{5})$

SOLUTIONS TO APPLICATIONS PROBLEMS:

1. Let $\theta = \text{Arc sin } \frac{1}{2}$ $\quad\theta$ is the angle whose sine is $\frac{1}{2}$.
   $\therefore \sin\theta = \frac{1}{2}$
   $\therefore \quad \theta = 30^\circ$ or $\frac{\pi}{6}$ radians. (principal value).

2. Let $\theta = \text{Arc cos } \frac{1}{2}$
   $\cos\theta = \frac{1}{2}$
   $\theta = 60^\circ$ or $\frac{\pi}{3}$ radians.

3. $\theta = \text{Arc tan } (-1)$
   $\tan\theta = -1$
   $\theta = -45^\circ$ or $-\frac{\pi}{4}$ radians.

4. $\text{Arc sin } 2 = \theta.$
   $\sin\theta = 2$
   has no solution since
   $-1 \leq \sin\theta \leq 1$
   Thus Arc sin 2 is undefined.

5. $\theta = \text{Arc}\cos 0$

$\cos\theta = 0$

$\theta = 90^{\circ}$ or $\frac{\pi}{2}$ radians.

6. $\sin(\text{Arc}\tan 1)$

Let $\theta = \text{Arc}\tan 1$

$\tan\theta = 1$

$\theta = 45^{\circ}$

$\therefore \sin\theta = \sin 45^{\circ} = \frac{\sqrt{2}}{2}$

$\therefore \sin(\text{Arc}\tan 1) = \frac{\sqrt{2}}{2}$

7. $\tan(\text{Arc}\cos \frac{5}{13})$

Let $\theta = \text{Arc}\cos \frac{5}{13}$

$\therefore \cos\theta = \frac{5}{13}$

$\therefore \tan\theta = \frac{12}{13}$

$\therefore \tan(\text{Arc}\cos \frac{5}{13}) = \frac{12}{13}$

8. $\cot(\text{Arc}\tan \frac{7}{3})$

$\theta = \text{Arc}\tan \frac{7}{3}$

$\tan\theta = \frac{7}{3}$

So $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{1}{\frac{7}{3}} = \frac{3}{7}$.

$\therefore \cot(\text{Arc}\tan \frac{7}{3}) = \frac{3}{7}$

9. $\cos 2(\text{Arc}\sin \frac{-3}{5})$

$\theta = \text{Arc}\sin \frac{-3}{5}$

$\sin\theta = \frac{-3}{5}$ and θ must be in quadrant IV.

$\therefore \cos 2\theta = 1 - 2\sin^2\theta$

$= 1 - 2\cdot(\frac{-3}{5})^2 = 1 - 2\cdot\frac{9}{25}$

$\cos 2\theta = \frac{7}{25}$

$\therefore \cos 2(\text{Arc}\sin \frac{-3}{5}) = \frac{7}{25}$

SUMMARY:

Express $\cos A = \frac{3}{5}$ in inverse trigonometric form.

How do we evaluate an expression containing an inverse trigonometric function?

1. Introduce an angle (say, θ)
2. Draw a right triangle in the proper quadrant, if necessary. Take principal value into account.

What are the principal value restrictions for the sine function? For the cosine function? For the tangent function?

LESSON: 40*

* This lesson is optional and can be done in any honors class or in any class where time permits.

AIM: What are polar coordinates?

PERFORMANCE OBJECTIVES: The student will be able to...

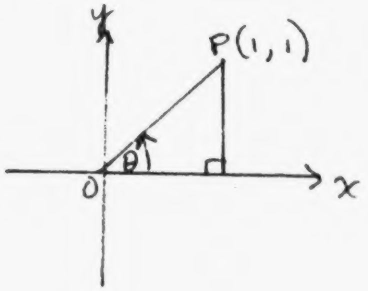
1. Graph points, given their polar coordinates.
2. State three different pairs of polar coordinates for a point given it's rectangular coordinates. Not all three pairs should have the same r coordinates.
3. Convert polar coordinates to rectangular coordinates and vice versa.
4. Sketch a simple curve, given its polar equation.

VOCABULARY: pole, polar axis, polar coordinates

MOTIVATION:

1. How far is P (1,1) from the origin, O?
2. What angle (in standard position) does $\overrightarrow{OP}$ determine with the positive portion of the x-axis?

Figure 1



$(OP)^2 = 1^2 + 1^2 = 2$

$OP = \sqrt{2}$

$\theta = \text{Arc tan } 1 = 45^\circ.$

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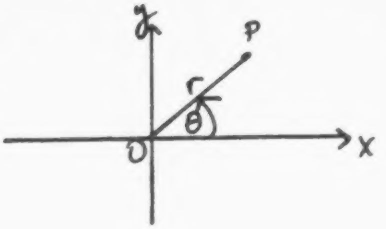
DEVELOPMENT:

Suppose that a point, P', is located $\sqrt{2}$ units from 0, what is a possible position for P'? Are any other positions possible for P'? Students will describe a circle with center, 0, and radius = $\sqrt{2}$.

What additional piece of information must we have to locate a particular point on the circle? A central angle.

Introduce the notation: (r, θ) to denote the polar coordinates of a point P as in the following diagram:

Figure 2



O is the pole.
The positive portion of the x-axis is the polar axis.

In fig. 1, draw a circle with radius = OP = $\sqrt{2}$.
Have students determine the polar coordinates of each point where the circle intersects an axis. Label them Q, R, S, and T starting with Q on the positive x-axis. Label them Q, R, S and T starting with Q on the positive x-axis and proceeding counterclockwise.

Ask students to plot the point whose polar coordinates are:

1. $(\sqrt{2}, -\pi)$
2. $(\sqrt{2}, \frac{-\pi}{2})$
3. $(\sqrt{2}, \frac{9\pi}{4})$
4. $(-\sqrt{2}, -\pi)$
5. $(\sqrt{2}, 0)$
6. $(-\sqrt{2}, \frac{9\pi}{4})$
7. $(0, \frac{\pi}{2})$

Problems 1 and 2 will recall clockwise notations. Problem 3 will recall that an angle of θ radians is $\cong$ to an angle of $\theta + 2k\pi$ radians where k is any integer. Problems 4, 5, and 6 introduce the idea of a directed distance and raise the possibility that a point can have more than one set of polar coordinates.

APPLICATIONS:

Express each pair of rectangular coordinates in polar form:

1. $(2, 2\sqrt{3})$ 2. $(-3, 3)$ 3. $(-5\sqrt{3}, 5)$

Express each pair of polar coordinates in rectangular form:

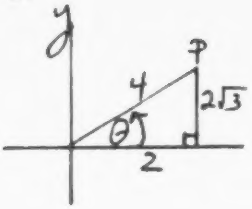
4. $(4, 60^{\circ})$ 5. $(5, 300^{\circ})$ 6. $(9, 720^{\circ})$

Graph:

7. $\theta = \pi$ 8. $r = 2$ 9. $r = 3\sin\theta$

SOLUTIONS TO APPLICATION PROBLEMS:

1.

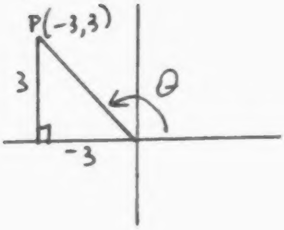


$r^2 = 2^2 + (2\sqrt{3})^2 = 4 + 12 = 16$; $r = 4$.

$\tan\theta = \frac{2\sqrt{3}}{2} = \sqrt{3}$; $\theta = 60^{\circ}$

$(4, 60^{\circ})$ is the polar form of $(2, 2\sqrt{3})$

2.

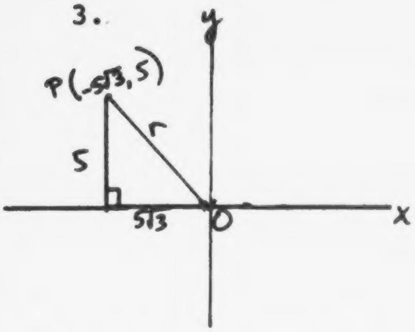


$r^2 = 3^2 + 3^2 = 18$; $r = 3\sqrt{2}.$

$\tan(180 - \theta) = \frac{3}{3} = 1$; $180 - \theta = 45^{\circ}$; $\theta = 135^{\circ}$

$(3\sqrt{2}, 135_{o})$ is the polar form of $(-3,3)$

3.



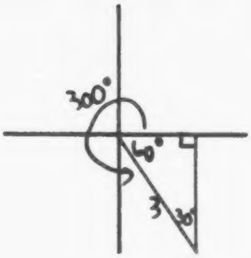
$r^2 = 5^2 + (5\sqrt{3})^2 = 100$; $r = 10$.

$180 - \theta = 30^\circ$; $\theta = 150^\circ$;

$(10, 150^\circ)$ is the polar form of $(-5\sqrt{3}, 5)$

4. See Problem 1: $(2, 2\sqrt{3})$

5.



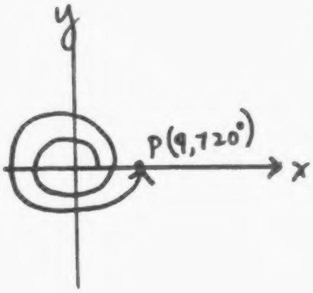
Right triangle is a dilation (scale factor of 3) of another one with OP = 1 and P $(\cos 60^\circ, \sin 60^\circ)$

$\therefore$ P' $(3\cos 60^\circ, 3\sin 60^\circ)$

P' $(3\cdot\frac{1}{2}, 3\cdot\frac{\sqrt{3}}{2})$

P' $(\frac{3}{2}, \frac{3\sqrt{3}}{2})$

6.



(9, 0) is the answer.

7. $\theta = \pi$ is independent of r. I.e., no matter what value r takes, the angle θ is 180°. If $r > 0$, $\theta = \pi$ is a ray which is the negative portion of the x - axis. If $r < 0$, $\theta = \pi$ is a ray which is the positive portion of the x - axis. If $r = 0$, $\theta = \pi$ is the origin.
The union of these sets is a line (the x - axis.)

8. $r = 2$ is independent of θ. I.e., no matter what value is assigned to θ, $r = 2$. We get a circle with a radius of 2 and center at origin.

9. $r = 3\sin\theta$; $r = \frac{3y}{r}$

$$r^2 = 3y$$
$$x^2 + y^2 = 3y$$
$$x^2 + y^2 - 3y = 0$$
$$x^2 + y^2 - 3y + \frac{9}{4} = \frac{9}{4}$$
$$x^2 + \left(y - \frac{3}{2}\right)^2 = \left(\frac{3}{2}\right)^2$$

is a circle with center $(0, \frac{3}{2})$ and radius of $\frac{3}{2}$.

ALTERNATIVE SOLUTION:

θ	0	30°	45°	60°	90°	120°	135°	150°	180°
$\sin\theta$	0	.5	.71	.87	1	.87	.71	.5	0
$3\sin\theta$	0	1.5	2.13	2.61	3	2.61	2.13	1.5	0

SUMMARY:

What equations unite polar coordinates with the more familiar rectangular coordinates?

$x^2 + y^2 = r^2$ $\quad \tan\theta = \frac{y}{x}$ $\quad \sin\theta = \frac{y}{r}$ $\quad \cos\theta = \frac{x}{r}$.

Lesson: 41

AIM: What are the basics of probability? (A review.)

PERFORMANCE OBJECTIVES: The student will be able to...

1. Review terms: Random experiment, outcome, sample space, event, probability, complementary events.
2. Solve basic probability problems.
3. Find the probability of one event or the other in an experiment or game (mutually exclusive events, intersecting events).

VOCABULARY: random experiment, outcome, sample space, event, probability, complementary events, mutually exclusive events, intersecting events

MOTIVATION:

What is the probability that a single card drawn from a well-shuffled deck of 52 cards is: (a) an ace; (b) a face card; (c) and ace or a face card?

DEVELOPMENT:

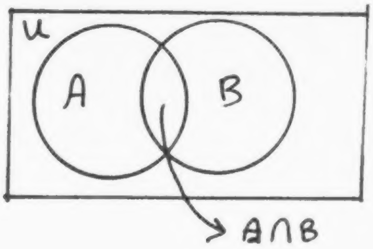
1. In order to solve problems such as the motivation problem, we must define basic terms. What are the definitions of: (a) sample space (set of all possible outcomes of a random experiment); (b) element (each individual outcome); (c) event (subset of the sample space); (d) U (the universal set), θ (the empty set), A' (complement of set A); (e) probability (to each element of the sample space, assign a probability, which is a positive number or zero. The sum of the probabilities assigned to all elements equals 1. $P(U)=1$. $P(U')=P(\theta)=0$. $0\leq P(A)\leq 1$. $P(A)+P(A')=1$).
2. What is the general strategy for solving probability problems? (a) list or describe the sample space. (Note that other methods of counting - using trees, combinations, or permutations, will be discussed in

future lessons). (b) Assign a probability to each element (uniform probability means that each element is assigned the same number). (c) List or describe the events (possibly complements of events). If applicable, use the Addition formula (the Multiplication formula will be discussed in a future lesson).

Addition formula for events A, B:

$$P(A \cup B) = P(A)+P(B)-P(A \cap B)$$

How would you interpret the Addition formula using a Venn diagram? Elicit the following diagram:



Note that $\cup$ means "or", $\cap$ means "and". The probability of an event being in $A \cup B$ (A or B) equals the probability of the event being in A plus the probability of the event being in B minus the probability of the event being in both A and B, since otherwise the event will be counted twice. If $A \cap B = \theta$, the events A and B are mutually exclusive. Then $P(A \cap B)=0$.

3. How is the general strategy used to solve the motivation problem? (a) The sample space is the set of 52 cards. (b) This is a uniform probability problem. The probability of choosing a card is $\frac{1}{52}$. (c) Event A = $\{4 \text{ aces}\}$. $P(A) = \frac{4}{52}$. Event B = $\{12 \text{ face cards}\}$ $P(B) = \frac{12}{52}$ (d) $P(A \cup B) = P(A)+P(B)-P(A \cap B) = \frac{4}{52} + \frac{12}{52} - 0 = \frac{4}{13}$
   Note that $P(A \cap B) = 0$ since A and B are mutually exclusive events. A card can not be both a face card and an ace.

4. Consider the following problem: What is the probability that a single card drawn from a well-shuffled deck of 52 cards is: (a) a king; (b) a face card; (c) a king or a face card?

5. How is the general strategy used to solve the previous problem? Steps (a) and (b) are the same as (3a) and (3b). (c) Event $A = \{4 \text{ kings}\}$. $P(A) = \frac{4}{52}$. Event $B = \{12 \text{ face cards}\}$. $P(B) = \frac{12}{52}$. (d) $P(A \cup B) = P(A)+P(B)-P(A \cap B) = \frac{4}{52} + \frac{12}{52} - \frac{4}{52} = \frac{3}{13}$. Note that A and B are intersecting events, since a King is a face card.

6. Using the same deck of cards, what is the probability that a single card is: (a) not a ace; (b) not a face card; (c) neither an ace nor a face card?

7. What is the general strategy for solving the previous problem? Steps (a) and (b) are the same as (3a) and (3b). (c) Event $A = \{4 \text{ aces}\}$. Event $A' = \{48 \text{ cards not an ace}\}$. $P(A') = \frac{48}{52} = \frac{12}{13}$. Event B = 12 face cards . Event $B' = \{40 \text{ cards not face cards}\}$. $P(B') = \frac{40}{52} = \frac{10}{13}$. (d) $P(A \cup B) = \frac{4}{13}$. Probability of neither an ace nor a face card = 1 - probability of a face card or an ace = $1 - \frac{4}{13} = \frac{9}{13}$.

* 8. A Venn diagram can be used to illustrate the following Addition formula for 3 events: $P(A \cup B \cup C) = P(A)+P(B)+P(C)-P(A \cap B)-P(A \cap C)-P(B \cap C)+P(A \cap B \cap C)$.
An example is: What is the probability a card is an ace or a king or a red card?
Let $A = \{4 \text{ aces}\}$, $B = \{4 \text{ kings}\}$, $C = \{26 \text{ red cards}\}$.
$P(A \cup B \cup C) = \frac{4}{52} + \frac{4}{52} + \frac{26}{52} - 0 - \frac{2}{52} - \frac{2}{52} + 0 = \frac{15}{26}$.

APPLICATIONS:

1. A number is selected at random from the set of positive integers from 1 to 12. What is the probability that the number chosen is either less than 8 or is odd?

2. A bag contains 9 red, 8 white, 6 blue, and 10 black marbles. If one marble is taken from the bag, what is the probability it is: (a) not black; (b) red or blue?

3. Two fair dice are rolled. What is the probability that the sum appearing on both dice is 7 or 11?

4. For a given year, assume the probability the Yankees will

win the pennant is $\frac{3}{7}$, the probability the Mets will win the pennant is $\frac{2}{5}$, and the probability both will win the pennant is $\frac{6}{35}$. What is the probability the Yankees or the Mets will win the pennant?

5. A coin is tossed and a pair of dice are rolled. What is the probability a head is tossed or a sum of 2 is rolled?

6. A class is going on a trip to Disney World. The probability the class goes by plane is $\frac{1}{3}$ and by train is $\frac{1}{2}$. What is the probability the class neither travels by plane nor by train?

SOLUTIONS TO APPLICATION PROBLEMS:

1. $\frac{7}{12} + \frac{6}{12} - \frac{4}{12} = \frac{9}{12} = \frac{3}{4}$

2a. $1 - \frac{10}{33} = \frac{23}{33}$

b. $\frac{9}{33} + \frac{6}{33} = \frac{15}{33} = \frac{5}{11}$

3. Sample set = $\{36 \text{ numbers}\}$
Event A = $\{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$
$P(A) = \frac{6}{36}$. Event B = $\{(5,6), (6,5)\}$. $P(B) = \frac{2}{36}$
$P(AUB) = \frac{6}{36} + \frac{2}{36} - 0 = \frac{8}{36} = \frac{2}{9}$

4. $P(AUB) = \frac{3}{7} + \frac{2}{5} - \frac{6}{35} = \frac{15+14-6}{35} = \frac{23}{35}$

5. $P(AUB) = \frac{1}{2} + \frac{1}{36} - 0 = \frac{19}{36}$

6. $P(AUB) = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$; $1 - \frac{5}{6} = \frac{1}{6}$

SUMMARY:

1. What are the definitions of: (a) sample space; (b) element; (c) event; (d) universal set; (e) probability?

2. What are the steps in the general strategy for solving probability problems?

3. What is the Addition formula for probability?

4. What are mutually exclusive events?

LESSON: 42

AIM: How do you use tree diagrams to find probabilities in experiments or games that have two or more stages (trials)?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Draw tree diagrams at each stage of an experiment, with the probabilities listed.
2. Determine the probability of any outcome in the sample space by multiplying probabilities along the path.
3. Determine the probability of events without replacement.
4. Determine the probability of events with replacement.

VOCABULARY: multistage events, tree diagrams, with or without replacement

MOTIVATION:

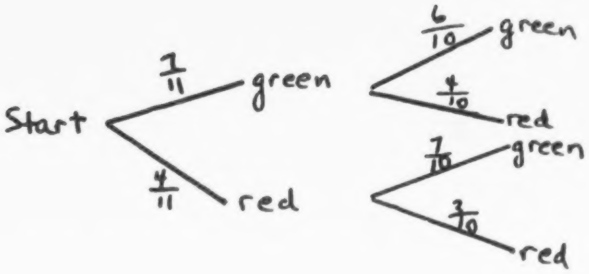
If 2 balls are drawn from an urn which contains 7 green and 4 red balls, what is the probability of the following events:

a) 2 green balls are drawn;
b) 1 green and 1 red ball are drawn in that order;
c) 1 green and 1 red ball are drawn;
d) 2 red balls are drawn?

DEVELOPMENT:

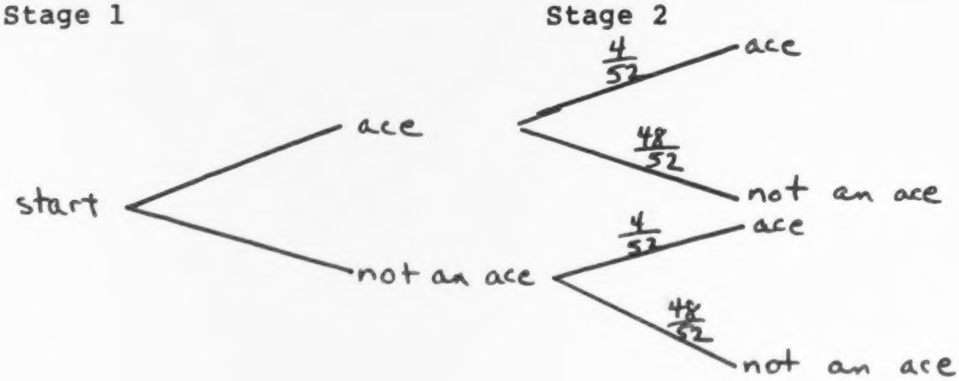
1. Explain to the class that tree diagrams are helpful for analyzing problems with multistage events, that is events following each other. The drawing of the first ball from the urn can be considered to be stage 1 and the drawing of the second urn to be stage 2. In a tree diagram, the probability should be indicated along each path. **To find the probability of any outcome in the sample space, MULTIPLY THE PROBABILITIES ALONG THE PATHS FROM THE START TO THE PARTICULAR OUTCOME.** (Note that this will be generalized as the Counting Principle in a future lesson). A tree diagram for the motivation problem is:

Stage 1 Stage 2



Note that this is a problem without replacement, since the first ball is not placed back in the urn. Unless stated otherwise, problems are assumed to be without replacement. In computing the probabilities for stage 2 you must consider what happened in stage 1; the number of balls decreased from 11 to 10 (hence the denominators are 10). As a check of the probabilities along the paths, notice that the sum of 2 probabilites for each section of each stage equals 1 ($\frac{7}{11} + \frac{4}{11} = 1$, $\frac{6}{10} + \frac{4}{10} = 1$, $\frac{7}{10} + \frac{3}{10} = 1$). For part (a) of the motivation problem: probability of 2 green balls = probability along path to stage 1 green times probability along path to stage 2 green = $\frac{7}{11} \cdot \frac{6}{10} = \frac{42}{110} = \frac{21}{55}$. (b) Probability of 1 green followed by 1 red = probability along path to stage 1 green times probability along path to stage 2 red = $\frac{7}{11} \cdot \frac{4}{10} = \frac{28}{110} = \frac{14}{55}$. (c) Probability of 1 green and 1 red = probability along path to stage 1 green times probability along path to stage 2 red plus (using the Addition formula) probability along path to stage 1 red times probability along path to stage 2 green = $\frac{7}{11} \cdot \frac{4}{10} + \frac{4}{11} \cdot \frac{7}{10} = \frac{56}{110} = \frac{28}{55}$. (d) Probability of 2 red balls = probability along path to stage 1 red times probability along path to stage 2 red = $\frac{4}{11} \cdot \frac{3}{10} = \frac{12}{110} = \frac{6}{55}$.

2. Consider a problem with replacement. Two cards are selected with replacement (that is, after a card is selected, it is put back into the deck) from a deck of 52 cards. What is the probability of the following events: (a) both cards are aces; (b) exactly one card is an ace; (c) neither card is an ace? Draw a tree diagram:



For part (a): Probability of 2 aces = $\frac{4}{52} \cdot \frac{4}{52} = \frac{1}{169}$. Part (b): Probability of 1 ace = $\frac{4}{52} \cdot \frac{48}{52} + \frac{48}{52} \cdot \frac{4}{52} = \frac{24}{169}$. Part (c): Probability of no aces = $\frac{48}{52} \cdot \frac{48}{52} = \frac{144}{169}$.

Note that for a problem with replacement, the denominator of the fraction for the probability (that is, the sample set of 52 cards) in stage 2 is the same as for stage 1.

APPLICATIONS:

1. A lottery is held for a high school graduating class, which consists of 60 girls and 40 boys. The first drawing is for a $1,000 prize and the second drawing is for a $500 prize. A student may not win both prizes. What is the probability that: (a) both prizes are won by boys; (b) the $1000 prize is won by a boy and the $500 prize is won by a girl; (c) the $1000 prize is won by a girl and $500 prize is won by a boy; (d) a boy wins one prize and a girl wins one prize?

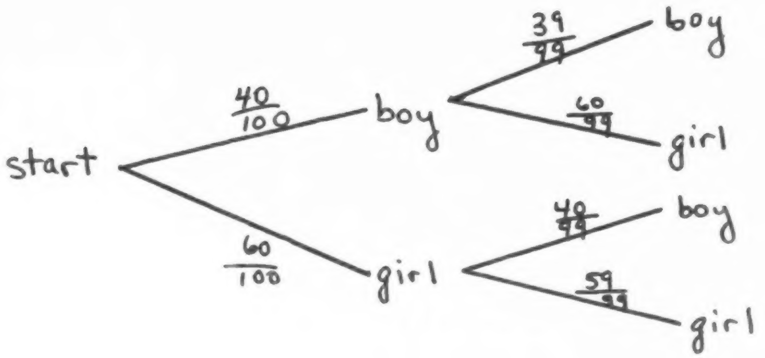
2. For Application problem (1), assume that a student may win both prizes. Answer parts (a) through (d).

3. Assume the probability of giving birth to a boy equals the probability of giving birth to a girl. What is the probability a family with two children has: (a) both girls; (b) a boy followed by a girl; (c) a girl followed by a boy; (d) one boy and one girl?

* 4. If 3 balls are drawn from an urn which contains 10 blue and 3 white marbles, what is the probability of the following events: (a) 3 blue marbles are drawn; (b) 2 blue marbles followed by 1 white marble are drawn; (c) 2 blue marbles and 1 white marble are drawn?

SOLUTIONS TO APPLICATION PROBLEMS:

1. Stage 1 ($1000) Stage 2 ($500)



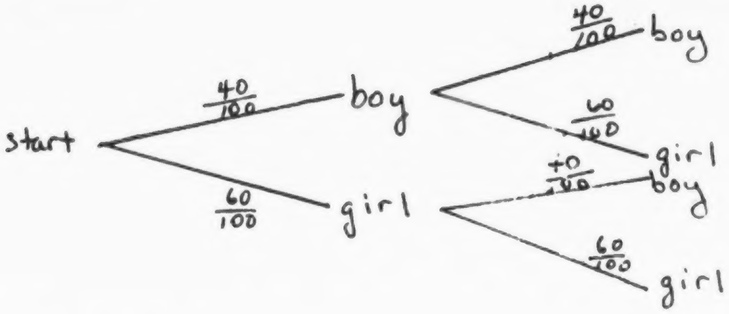
a) $\frac{40}{100} \cdot \frac{39}{99} = \frac{26}{165}$

b) $\frac{40}{100} \cdot \frac{60}{99} = \frac{8}{33}$

c) $\frac{60}{100} \cdot \frac{40}{99} = \frac{8}{33}$

d) $\frac{8}{33} + \frac{8}{33} = \frac{16}{33}$

2. Stage 1 ($1000) Stage 2 ($500)


start
40/100
60/100
boy
girl
40/100
boy
60/100
girl
40/100
boy
60/100
girl

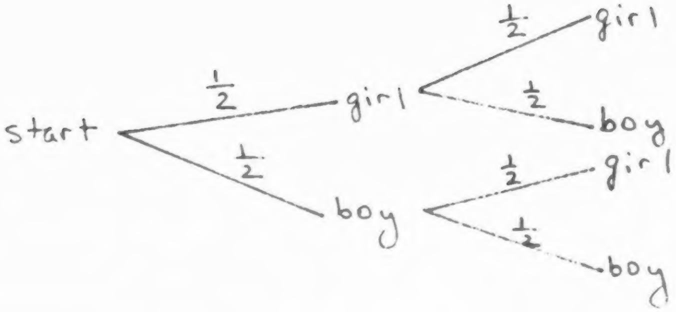
a) $\frac{40}{100} \cdot \frac{40}{100} = \frac{4}{25}$

b) $\frac{40}{100} \cdot \frac{60}{100} = \frac{6}{25}$

c) $\frac{60}{100} \cdot \frac{40}{100} = \frac{6}{25}$

d) $\frac{6}{25} + \frac{6}{25} = \frac{12}{25}$

3. Stage 1 Stage 2


start
1/2
1/2
girl
boy
1/2
girl
1/2
boy
1/2
girl
1/2
boy

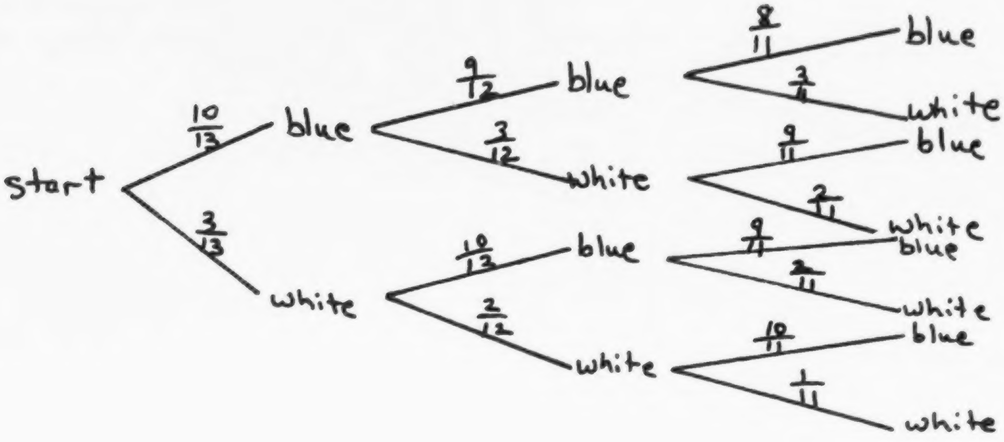
a) $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

b) $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

c) $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

d) $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

4. Stage 1 Stage 2 Stage 3


start
10/13
blue
3/13
white
9/12
blue
3/12
white
10/13
blue
2/12
white
8/11
blue
3/11
white
9/11
blue
2/11
white
9/11
blue
2/11
white
10/11
blue
1/11
white

a) $\frac{10}{13} \cdot \frac{9}{12} \cdot \frac{8}{11} = \frac{60}{143}$

b) $\frac{10}{13} \cdot \frac{9}{12} \cdot \frac{3}{11} = \frac{45}{286}$

c) $\frac{45}{286} + \frac{10}{13} \cdot \frac{3}{12} \cdot \frac{9}{11} + \frac{3}{13} \cdot \frac{10}{12} \cdot \frac{9}{11} = \frac{135}{286}$

SUMMARY:

1. What are multistage events?
2. How do you set up a tree diagram for multistage events?
3. What are the differences in probabilities along the paths between problems with replacement and problems without replacement?
4. How do you use the tree diagram to determine the probabilities of outcomes?

LESSON: 43

AIM: What is the probability of an event occurring on the condition that another event already occurred?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Determine when a probability problem is a conditional probability problem.
2. Apply the formula for conditional probability, $P(A \cap B) = P(B) \cdot P(A/B)$.
3. Use a tree diagram to solve conditional probability problems.
4. Determine when events are independent or dependent.

VOCABULARY: conditional probability, independent events, dependent events

MOTIVATION:

Maria draws a card from a deck of 52 cards. You are told that the card is a picture card. What is the probability that the card is a queen?

DEVELOPMENT:

1. Explain to the class that the conditional probability of B, given A (symbolized as P(B/A)), means the probability that event B will occur given that event A has already occurred. The general rule is·

   $$P(B/A) = \frac{P(A \cap B)}{P(A)} \qquad (P(A \cap B) \text{ means } P(A \text{ and } B))$$

   Since event A has already occurred, it is the "new" sample space (hence P(A) is the denominator) for P(B/A). For the motivation problem, let event A = $\{$12 picture cards$\}$, event B = $\{$4 queens$\}$.

   $$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{4}{52}}{\frac{12}{52}} = \frac{1}{3}$$

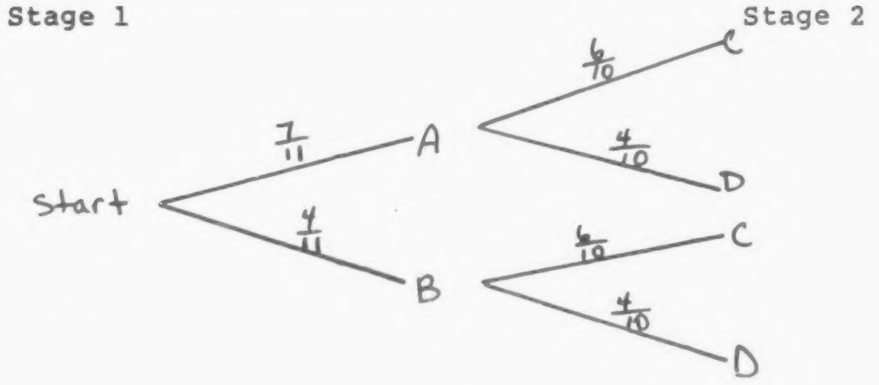
Note that $P(B) = \frac{4}{52}$. Since $P(B) = P(B/A)$, the probability of event B changed since event A occurred. Therefore, events B and A are dependent events.

If $P(B) = P(B/A)$, then events B and A are independent events. The formula for conditional probability then becomes:

$P(B) = \frac{P(A \cap B)}{P(A)}$ or $P(B) \cdot P(A) = P(A \cap B)$,

Which is the multiplication formula for probability ("and" means multiply).

2. Consider again the motivation problem from the previous lesson, that 2 balls are drawn from an urn which contains 7 green and 4 red balls. By using a tree diagram to calculate the probabilities, we actually used the principles of conditional probability. Let the events be: A - the first ball is green; B - the first ball is red; C - the second ball is green; D - the second ball is red. The tree diagram is:



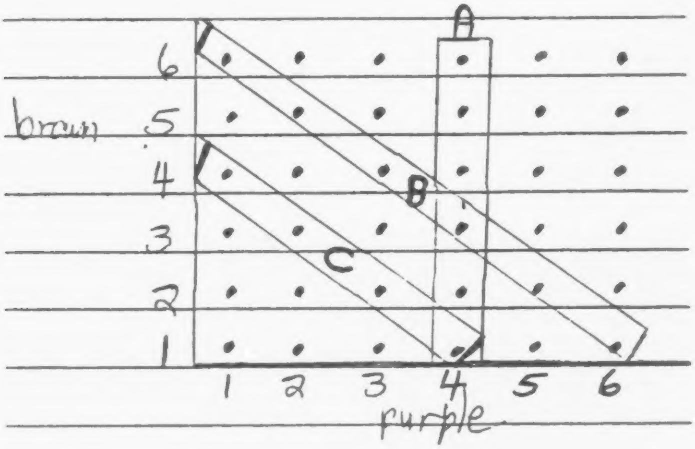
a) $P(\text{2 greens}) = P(A) \cdot P(C/A) = \frac{7}{11} \cdot \frac{6}{10} = \frac{21}{55}$

b) $P(\text{green followed by red}) = P(A) \cdot P(D/A) = \frac{7}{11} \cdot \frac{4}{10} = \frac{14}{55}$

c) $P(\text{green and red}) = P(A) \cdot P(D/A) + P(B) \cdot P(C/B) = \frac{7}{11} \cdot \frac{4}{10} + \frac{4}{11} \cdot \frac{7}{10} = \frac{28}{55}$

d) $P(\text{2 reds}) = P(B) \cdot P(D/B) = \frac{4}{11} \cdot \frac{3}{10} = \frac{6}{55}$

3. A brown die and a purple die are rolled. Let the events be: A - purple die shows 4; B - sum of the numbers is 7; C - sum of the numbers is 5. (a) Are events A and B independent? (b) Are events A and C independent? (c) Are events B and C independent? While tree diagrams are helpful for problems with urns, cards, and births, charts are helpful for problems with dice.



$P(A) = \frac{6}{36}$ $P(B) = \frac{6}{36}$ $P(C) = \frac{4}{36}$ $P(A \cap B) = \frac{1}{36}$

$P(A \cap C) = \frac{1}{36}$ $P(B \cap C) = 0$

(a) $P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{36}}{\frac{6}{36}} = \frac{1}{6}$

Since $P(B) = P(B/A) = \frac{1}{6}$, events A and B are independent.

(b) $P(C/A) = \frac{P(A \cap C)}{P(A)} = \frac{\frac{1}{36}}{\frac{6}{36}} = \frac{1}{6}$

Since $P(C) \neq P(C/A)$, events A and C are dependent.

(c) $P(C/B) = \frac{P(B \cap C)}{P(B)} = \frac{0}{\frac{6}{36}} = 0.$

Since $P(C) \neq P(C/B)$, events C and B are dependent. Note that intersecting events may be independent (A and B) or dependent (A and C), while mutually exclusive events with positive probabilities (B and C) are always dependent (since $P(B\ \ C) = 0$).

APPLICATIONS:

1. Of 100 cars in a showroom, 90 have air conditioning. Sixty of the air conditioned cars have automatic transmission, while 4 of the non-air conditioned cars have automatic transmission. If a car is picked at random, what is the probability that: (a) it has an automatic transmission, given that it has air conditioning; (b) it does not have automatic transmission, given that it doesn't have air conditioning?
2. Two dice are rolled and the sum is ≥ 9. What is the probability that the sum is: (a) 8; (b) 9 or 12; (c) an even number?
3. A card is drawn from a deck of 52 cards. Let the events be: A - card is red; B - card is a king; C- card is either a queen or a king. (a) Are events A and B independent? (b) Are events A and C independent? (c) Are events B and C independent?
4. A mauve die and a torquoise die are rolled. What is the probability that: (a) the mauve die shows a 5, given that the sum is even; (b) the torquoise die shows a 4, given that the sum is 8 or 9?

SOLUTIONS TO APPLICATION PROBLEMS:

1. Let the events be: A - air conditioning; B - no air conditioning; C - automatic transmission; D - no automatic transmission.

(a) $P(C/A) = \dfrac{P(A \cap C)}{P(A)} = \dfrac{\frac{60}{100}}{\frac{90}{100}} = \dfrac{2}{3}$ (b) $P(D/B) = \dfrac{P(B \cap D)}{P(B)} = \dfrac{\frac{4}{100}}{\frac{10}{100}} = \dfrac{2}{5}$

2. A chart might be helpful. Let the events be: A - sum is $\geq$ 9; B - sum is 8; C - sum is 9; D - sum is 12; E - sum is even. (a) $P(B/A) = \frac{P(A \cap B)}{P(A)} = 0$

(b) $P\ (C/A) + P(D/A) =$

$$\frac{P(A \cap C)}{P(A)} + \frac{P(A \cap D)}{P(A)} = \frac{\frac{(4\ )}{(36)}}{\frac{(10)}{(36)}} + \frac{\frac{(1\ )}{(36)}}{\frac{(10)}{(36)}} = \frac{1}{2}$$

(c) $P(E/A) = \frac{P(A \cap E)}{P(A)} = \frac{\frac{(4\ )}{(36)}}{\frac{(10)}{(36)}} = \frac{2}{5}$

3. $P(A) = \frac{26}{52} \qquad P(B) = \frac{4}{52} \qquad P(C) = \frac{8}{52} \qquad P(A \cap B) = \frac{2}{52}$

$P(A \cap C) = \frac{4}{52} \qquad P(B \cap C) = \frac{4}{52}$

(a) $P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{(2\ )}{(52)}}{\frac{(26)}{(52)}} = \frac{1}{13} = P(B)$ independent.

(b) $P(C/A) = \frac{P(A \cap C)}{P(A)} = \frac{\frac{(4\ )}{(52)}}{\frac{(26)}{(52)}} = \frac{2}{13} = P(C)$ independent.

(c) $P(C/B) = \frac{P(B \cap C)}{P(B)} = \frac{\frac{(4\ )}{(52)}}{\frac{(4\ )}{(52)}} = 1 \neq P(C)$ dependent.

4. A chart might be helpful. Let the events be: A - sum is even; B - mauve die shows 5; C - sum is 8 or 9; D - torquoise die shows 4.

(a) $P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{(3\ )}{(36)}}{\frac{(18)}{(36)}} = \frac{1}{6}$

(b) $P(D/C) = \frac{P(C \cap D)}{P(C)} = \frac{\frac{(2\ )}{(36)}}{\frac{(9\ )}{(36)}} = \frac{2}{9}$

SUMMARY:

1. What is the definition of conditional probability?
2. What is the formula for conditional probability?
3. What types of problems are tree diagrams helpful for?
4. What type of problem is a chart helpful for?
5. What are dependent events?
6. What are independent events?

LESSON: 44

AIM: How does one use the Counting Principle and permutation formula to solve counting and probability problems?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Determine when to apply the Counting Principle and permutation formula.
2. Determine the number of different arrangements of n objects taken r at a time with or without repetitions.
3. Determine the number of arrangements of n objects when some of the objects are not distinguishable.

VOCABULARY: counting principle, permutation, factorial

MOTIVATION:

Often the most difficult part of solving probability problems is to count the number of elements in the sample space or event. Systematic methods of counting are needed. Consider the following problem: In how many ways can 3 students be seated in 3 seats arranged in a row?

DEVELOPMENT:

1. What method(s) do we have for systematically counting the sample set? Elicit that we can use a list or a tree. Explain to the class that we can also use **the Counting Principle - if one activity can occur in any of m ways and another activity can occur in any of n ways, then the total number of ways that both activities can occur is given by m·n.** Note that the first activity and ("and" means multiply) the second activity both occur. Ask: How can the Counting Principle be applied to the motivation problem? Elicit there are 3 possible students who may sit in the first seat. There are only two students left for the second seat and one student left

for the third seat. Therefore, there are $3 \cdot 2 \cdot 1 = 6$ ways. This is a problem without repetition, since students, of course, can't be repeated.

2. The arrangements in the motivation problem can be called permutations. $3 \cdot 2 \cdot 1 = 3!$ (the ! symbol means factorial). For a natural number n, $n>1$, $n! = n(n-1)\ldots(3)(2)(1)$. By definition, $1! = 1$ and $0! = 1$. If r is a natural number and $r \leq n$, ${}_nP_r$ is defined as n objects taken r at a time. ${}_nP_r = \frac{n}{(n-r)!} = n(n-1)\ldots(n-r+1)$. ${}_nP_n = n!$ For example, ${}_6P_2 = 6(6-1)\cdots(6-2+1) = 6 \cdot 5 = 30$. For the motivation problem, ${}_3P_3 = 3! = 6$.

3. How may different 3 letter nonsense words can be made using 5 different letters with:
   a) No repetition of letters? Use the Counting Principle to get $5 \cdot 4 \cdot 3 = 60$ words, or use the permutation formula, ${}_5P_3 = 5 \cdot 4 \ldots (5-3+1) = 60$ words.
   b) Repetition of letters? Use the Counting Principle to get $5 \cdot 5 \cdot 5 \cdot = 125$ words. Note: For counting problems in which there is an arrangement of objects (that is, the order of the objects is important) use the Counting Principle or permutation formula if there is no repetition of objects and the Counting Principle if there is repetition of objects.

4. Out of 10 different pictures available to be hung, 5 are to be hung in a row. In how many different ways can these 5 pictures be hung if one particular picture out of the 10 must be hung first. Since there is no repetition of pictures, the problem may be solved with the Counting Principle or permutation formula. Using the Counting Principle, the answer is $1 \cdot 9 \cdot 8 \cdot 7$ (since there is only one choice for the first position) or 504. Using the permutation formula, the answer is $1 \cdot {}_9P_3 = 1 \cdot 9 \cdot 8 \cdots (9-3+1)$ or 504.

5. How many 4 letter nonsense words are there using the letters of the word roof. The answer is $\frac{{}_4P_4}{2}$, since there are two letters the same. Assume n objects have n_1 objects of one type, n_2 objects of a second type, n_3 objects of a third type and so on. The number of distinguishable permutations of the n objects is:

$$\frac{n!}{(n_1)!(n_2)!(n_3)!}$$

6. How many different 11 letter nonsense words can be formed using the letters of the word Mississippi? There are 4 s's, 4 i's, and 2 p's. The answer is:

$$\frac{11!}{4!\ 4!\ 2!}$$

APPLICATIONS:

1. In how many different ways can 9 people on a New York City subway train be seated:
   a) on a bench which seats 9 people in a row;
   b) if there is only room for 6 people on the bench?
2. In how many different ways can a student choose one history class and one foreign language class from 8 history electives and 7 foreign language electives?
3. Assume 4 books have black covers, 5 books have green covers, and 2 books have blue covers. In how many different ways can the books be arranged in a row so that books of the same color are kept together?
4. How many different license plates can be formed from 3 letters followed by 4 numbers?

* 5. Seven people are seated randomly. What is the probability that a designated two people sit next to each other if the seven people:
   a) sit in a row;
   b) sit at a circular table?

6. Using the digits 0,1,2,...9, how many numbers are there between 4000 and 6000 in which there is no repetition of digits?

SOLUTIONS TO APPLICATION PROBLEMS:

1. a) ${}_9P_9 = 9! = 362{,}880$
   b) ${}_9P_6 = 9 \cdot 8 \ldots (9-6+1) = 60{,}480$

2. $8 \cdot 7 = 56$

3. The books can be arranged in (4!)(5!)(2!) ways if the black covered books were followed by the green covered and then blue covered books. Since the colors may be permuted, (4!)(5!)(2!) is multiplied by (3!) to give 34,560 ways.

4. Since there can be repetition of letters and numbers (unless told otherwise), there are $26^3 \cdot 10^4$ different license plates.

5. a) The sample set has 7! ways. The number of ways that the two designed individuals may sit together is $2! \cdot 6!$ (since the two designated people may be permuted and there are 5 single people and one pair to be permuted). Probability = $\frac{2!6!}{7!} = \frac{2}{7}$.
   b) Sample set (in a circle) has 6! ways (since there are no endpoints). The event has 2!5! ways. Probability = $\frac{2!5!}{6!} = \frac{1}{3}$.

6. Note that 4000 and 6000 aren't counted, since there is repetition of digits. Since the first digit can be 4 or 5, there are two possibilities for the first digit. There are $2 \cdot {}_9P_3$ or 1008 different numbers.

SUMMARY:

1. What is the statement of the Counting Principle?

2. What is the formula for permutations of n objects taken r at a time?

3. For which types of problems are the Counting Principle and permutations used for?

4. In how many ways can n people be seated in a circle?

LESSON: 45

AIM: How does one use the combinations formula to solve counting and probability problems?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Compute combinations of n objects taken r at a time.
2. Find probabilities using combinations.
3. Determine whether a given counting or probability problem involves permutations or combinations.

VOCABULARY: combinations, choose notation, symbolic form

MOTIVATION:

In how many different ways can a subcommittee of three people be chosen from a committee of four people?

DEVELOPMENT:

1. Let the people be labelled a,b,c,d. Elicit from the class that the number of ways of choosing 3 people from $\{a,b,c,d\}$ is 4 $\{a,b,c\}$, $\{a,b,d\}$, $\{a,c,d\}$, $\{b,c,d\}$. A combination is a set of objects. Combinations are used for problems in which order does not count. Note that the set $\{a,b,c\}$ is the same subset of the set of $\{a,b,c,d\}$ as $\{c,b,d\}$. For the motivation problem, each of the 3! ways of arranging or permuting three letters only counts once for a combination. Using choose notation, let ${}_4C_3$ represent the combination of 4 objects taken 3 at a time. ${}_4C_3 = \frac{{}_4P_3}{3!} = \frac{24}{6} = 4$. For nonnegative integers n and r, where $n \geq r$, let ${}_nC_r$ or $\binom{n}{r}$ represent the combination of n objects taken r at a time.

$${}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!} \quad \text{or} \quad \frac{n(n-1)\ldots(n-r+1)}{r!}$$

Note that ${}_nC_r = {}_nC_{n-r}$.

2. From a school board consisting of 12 members, how many different: (a) 3 member committees can be formed? ${}_{12}C_3 = \frac{12\cdot 11...(12-3+1)}{3!} = 220$; (b) committees consisting of a president, vice-president, and treasurer can be formed? Since the order of the members in the committee determines the rank, use a permutation.
   ${}_{12}P_3 = 12\cdot 11...(12-3+1) = 1320$.

3. If 5 cards are drawn from a deck of 52 cards, what is the probability that there are exactly 3 cards that are queens? Since the order of the cards is not important, combinations should be used. The sample set has ${}_{52}C_5$ possible hands. For the event of 3 queens and 2 cards that are not queens, the 3 queens are chosen from 4 cards and the other 2 cards are chosen from the remaining 48 cards. Using the Counting Principle, there are ${}_4C_3 \cdot {}_{48}C_2$ possible hands in the event. Probability =

   $\frac{{}_4C_3 \cdot {}_{48}C_2}{{}_{52}C_5}$ (in symbolic form) or $\frac{4512}{2{,}598{,}960}$.

4. Note that for problems involving numbers or words, where order is important, permutations are used. For problems involving subsets, such as of people, socks, or marbles, where order is not important, combinations are used.

APPLICATIONS:

1. In how many different ways can: (a) a subcommittee of 4 Democrats and 2 Republicans be chosen from 10 Democrats and 12 Republicans; (b) 4 brown socks and 2 blue socks be chosen (in the dark) from a drawer with 10 brown socks and 12 blue socks?

2. How many distinct lines can be drawn which contain 2 out of 6 points in a plane? Assume that no 3 points are collinear.

3. How many diagonals are there in an octagon (8 sided polygon)?

4. How many different license plates can be formed from 3 letters and 4 numbers, if the letters and numbers can be arranged in any order?

* 5. At a party there are 66 handshakes. Assume that each person shakes hands with everyone else at the party exactly once. How many people are at the party?

6. An urn has 9 red, 7 green, and 6 brown marbles. If 3 marbles are taken from the urn, what is the probability that the marbles have the same color (in symbolic form)?

7. In a lottery game, 6 numbers are chosen from the integers 1 to 48. What is the probability that a person who chooses 6 numbers will choose the winning 6 numbers (in symbolic form)?

SOLUTIONS TO APPLICATION PROBLEMS:

1. a) ${}_{10}C_4 \cdot {}_{12}C_2 = 210 \cdot 66 = 13860$
   b) 13860

2. Since the order of the points for a line doesn't matter, there are ${}_6C_2$ or 15 lines.

3. ${}_8C_2 - 8$ (the number of sides) = 20 diagonals.

4. There are ${}_7C_3$ (or ${}_7C_4$) ways of choosing the order of the letters (or numbers). There are $26^3 \cdot 10^4 \cdot {}_7C_3$ different license plates.

5. $\binom{n}{2} = 66$. $\frac{n(n-1)}{2} = 66$. $n^2 - n = 132$. $n^2 - n - 132 = 0$.
   $(n-12)(n+11) = 0$. $n = 12$ people.

6. Sample set $= {}_{22}C_3$. Event = 3 red or 3 green or 3 brown $= {}_9C_3 + {}_7C_3 + {}_6C_3$.

$$\text{Probability} = \frac{{}_9C_3 + {}_7C_3 + {}_6C_3}{{}_{22}C_3}$$

7. $$\text{Probability} = \frac{1}{{}_{48}C_6}$$

SUMMARY:

1. What is a combination?

2. What is the formula for ${}_nC_r$?

3. What is the difference in the types of problems requiring combinations and permutations?

LESSON: 46

AIM: What is the probability of a particular event occurring in a sequence of independent trials of an experiment?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Recognize random experiments as Bernoulli experiments.
2. Compute successes in n trials by using the formula ${}_nC_rX^rY^{n-r}$ (Where X is the probability of success, Y=1-X).

VOCABULARY: Bernoulli experiments

MOTIVATION:

There are four true-false questions on a quiz. What is the probability that a student who guesses randomly will get exactly 1 question correct?

DEVELOPMENT:

1. Elicit from the class that the probability of getting a correct answer, P(C), equals the probability of getting an incorrect answer. $P(I)=P(C)=\frac{1}{2}$. Assume the correct answer is first. Elicit using the Counting Principle, $P(CIII) = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2}^4$. There are ${}_4C_1$ or 4 ways of choosing the correct answer (as the 1st, 2nd, 3rd, or 4th question). Therefore, the probability of getting exactly 1 question correct is ${}_4C_1\cdot\left(\frac{1}{2}\right)^4 = \frac{1}{4}$.

2. Explain to the class that a multistage probability experiment, such as the motivation problem, which consists of independent trials with one particular event, is called a Bernoulli experiment. Let the probability of event A' (for example, choosing a correct answer) be r. The probability of event A' (choosing an incorrect answer) is 1-r. Then the probability that event A occurs exactly r times is ${}_nC_rp^r(1-p)^{n-r}$. This formula applies to problems with two possibilities, sometimes thought of

as success and failure. Examples of such problems are true-false questions, tosses of a coin, births, and wins or losses in sports events.

3. Assume that Darryl is a .300 hitter in baseball, that is that the probability of him getting a hit is .3 and getting an out is .7. In his next 5 official times at bat (counting only hits and outs), what is the probability that Darryl gets:
   a) exactly 2 hits? Elicit: Let A = event of a hit. P(A) = .3. A' = event of an out. P(A') = .7. ${}_nC_r$ = ${}_5C_2$ (2 hits out of 5). The probability of exactly 2 hits is ${}_5C_2\cdot(.7)^3\cdot(.3)^2 = .3087$;
   b) at least 3 hits? Elicit: The probability of at least 3 hits equals the probability of exactly 3 hits plus the probability of exactly 4 hits plus the probability of exactly 5 hits = ${}_5C_3\cdot(.7)^2\cdot(.3)^3 + {}_5C_4\cdot(.7)^1\cdot(.3)^4 + {}_5C_5\cdot(.3)^5 = .1323+.02835+.00243 = .16308$. Note that the sum of the exponents of each term is always 5.

APPLICATIONS:

1. Assume the SAT English section has 60 questions, with each question having 5 choices. What is the probability that a person who does not understand English gets a perfect score by randomly guessing (give answer in symbolic form)?

2. Assume John beats Jimmy in tennis 3 out of 5 times. In the next 4 sets, what is the probability that John wins exactly 3 of the sets?

3. Assume that the probability of having a boy equals the probability of having a girl. In a family with 7 children, what is the probability that there are at least 5 girls (give answer in symbolic form).

4. A die is rolled 5 times. What is the probability of getting exactly three 4's (give answer in symbolic form)?

5. There are 7 green and 3 blue marbles in an urn. What is the probability of getting exactly 2 green balls if 5 balls are drawn: (a) with replacement; (b) without replacement?

SOLUTIONS TO APPLICATION PROBLEMS:

1. ${}_{60}C_{60}(.2)^{60}$ or $(.2)^{60}$

2. P(John winning) = .6. P(Jimmy winning) = .4.
   ${}_4C_3(.4)^1(.6)^3 = .3456$.

3. ${}_7C_5(.5)^2(.5)^5+{}_7C_6(.5)^1(.5)^6+{}_7C_7(.5)^7$

4. ${}_5C_3\left(\frac{1}{6}\right)^2\left(\frac{1}{6}\right)^3$

5. a) This is a Bernoulli experiment, since by replacing the marbles each trial is independent. Probability (green) = .7. Probability(blue) = .3. Probability of exactly 2 green balls = ${}_5C_2(.3)^3\cdot(.7)^2 = .1323$.

   b) Use combinations because the order is not important. Sample space = ${}_{10}C_5$. Event of 2 green, 3 blue marbles = ${}_7C_2\cdot{}_3C_3$ (using Counting Principle).

   Probability = $\dfrac{{}_7C_2 \cdot {}_3C_3}{{}_{10}C_5} = \dfrac{1}{12}$

SUMMARY:

1. What is a Bernoulli experiment?

2. What formula can be used to solve Bernoulli experiments?

3. What types of problems are Bernoulli experiments?

LESSON: 47

AIM: How can the binomial theorem be used to solve probability problems?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Use the binomial theorem to expand binomials.
2. Determine a certain term of an expanded binomial by the binomial theorem.
3. Use Pascal's triangle to determine coefficients of the binomial expansion.
4. Use the binomial theorem to solve Bernoulli experiment problems.

VOCABULARY: binomial theorem, binomial coefficients, Pascal's triangle

MOTIVATION:

1. Assume the probability of having a boy equals the probability of having a girl. In a family with 3 children, what is the probability of having: (a) no girls; (b) 1 girl; (c) 2 girls; (d) 3 girls?
2. Expand $(x+y)^3$.

DEVELOPMENT:

1. Elicit that the motivation problem (1) is a Bernoulli experiment. Therefore, the probabilities of getting from 0 to 3 girls is, respectively, ${}_3C_0\left(\frac{1}{2}\right)^3$, ${}_3C_1\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^1$, ${}_3C_2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2$, ${}_3C_3\left(\frac{1}{2}\right)^3$. The coefficients are 1,3,3,1. The expansion of $(x+y)^3$ is $1\cdot x^3 + 3x^2y^1 + 3xy^2 + 1\cdot y^3$. The reason that $(x+y)^3$ can be written as ${}_3C_0x^3 + {}_3C_1x^2y^1 + {}_3C_2xy^2 + {}_3C_3y^3$ is that the coefficients represent the number of ways one can choose the requisite number of x's (0,1,2, or 3) from the product $(x+y)\cdot(x+y)\cdot(x+y)$. Therefore, expansions of

binomials (such as x+y) can be used to solve Bernoulli experiment problems. The expansion of $(x+y)^3$ illustrates the Binomial theorem:

For positive integer n,

$$(x+y)^n = {}_nC_0x^n + {}_nC_1x^{n-1}y^1 + \ldots + {}_nC_{r-1}x^{n-(r-1)}y^{r-1} + \ldots + {}_nC_ny^n.$$

Coefficients ${}_nC_r$, where r=0,1,2,...n are called binomial coefficients. Note that the exponent of y with a coefficient of ${}_nC_r$ is r. For the rth term, the binomial coefficient is ${}_nC_{r-1}$ or $\binom{n}{r-1}$.

2. Use the Binomial theorem to expand $(x+y)^5$.

$$(x+y)^5 = {}_5C_0x^5 + {}_5C_1x^4y^1 + {}_5C_2x^3y^2 + {}_5C_3x^2y^3 + {}_5C_4xy^4 + {}_5C_5y^5 = x^5 + 5x^4y^1 + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$$

Note: The sum of the exponents of each term always equals n. Notice the symmetry of the coefficients (1,5,10,10,5,1).

3. Pascal's triangle provides an alternate method (especially for smaller values of n) for computing the binomial coefficients. The first 5 rows of Pascal's triangle are:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

In each row, the first and last entries are 1. The other numbers in the row are obtained by adding the numbers to the immediate left and right in the previous row. For example, the sixth row is obtained from the fifth row as

1 4 6 4 1
follows: 1 5 10 10 5 1 Note that the numbers 1,5,10,10,5,1 are the coefficients of the expansion of $(x+y)^5$.

4. Find the fourth term of the expansion of $(a+3b)^6$. Let x=a, y=3b. (a) Use Pascal's triangle: Find the seventh

1 5 10 10 5 1
row - 1 6 15 20 15 6 1

The coefficient of the fourth term is 20. The fourth term is $20x^3y^3$. Using the original letters, the answer is $20a^3(3b)^3$ or $540a^3b^3$. (b) Use the Binomial theorem: For the fourth term, r=4. The fourth term is ${}_6C_{4-1}x^3y^3$ or ${}_6C_3x^3y^3$, which equals $20x^3y^3$. Again, $20x^3y^3$ is $540a^3b^3$.

APPLICATIONS:

Express answers in symbolic form.

1. Assume Marina beats Chris in tennis 5 out of 7 times. In the next 12 sets, what is the probability that Chris wins exactly 8 sets?
2. Expand: $(a-2b)^8$.
3. What is the coefficient of x^4y^5 in the expansion of $(x+4y)^9$.
4. Let a fair coin be tossed 10 times. What is the probability that there are at least 7 heads?
5. Find the term in the expansion of $\left(4c - \frac{3}{c}\right)^{14}$ that is a constant.

SOLUTIONS TO APPLICATION PROBLEMS:

1. P(Marina wins) = $\frac{5}{7}$. P(Chris wins) = $\frac{2}{7}$. Probability that Chris wins exactly 8 sets is $\binom{12}{8}\left(\frac{5}{7}\right)^4\left(\frac{2}{7}\right)^8$.

2. Let x = a, y = -2b. $(x+y)^8 = \binom{8}{0}x^8 + \binom{8}{1}x^7y^1 + \binom{8}{2}x^6y^2 + \binom{8}{3}x^5y^3 + \binom{8}{4}x^4y^4 + \binom{8}{5}x^3y^5 + \binom{8}{6}x^2y^6 + \binom{8}{7}x^1y^7 + \binom{8}{8}y^8$. Substituting for x and y, $(a-2b)^8 = \genfrac{}{}{0pt}{}{8}{0}a^8 + \binom{8}{1}a^7(-2b)^1 + \binom{8}{2}a^6(-2b)^2 + \binom{8}{3}a^5(-2b)^3 + \binom{8}{4}a^4(-2b)^4 + \binom{8}{5}a^3(-2b)^5 + \binom{8}{6}a^2(-2b)^6 + \binom{8}{7}a(-2b)^7 + \binom{8}{8}(-2b)^8$

3. For $\binom{9}{5}x^5(4y)^5$, the coefficient is $\binom{9}{5}\cdot 4^5$.

4. P(head) = P(tail) = $\frac{1}{2}$. Probability of at least 7 heads = $\binom{10}{7}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^7 + \binom{10}{8}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^8 + \binom{10}{9}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^9 + \binom{10}{10}\left(\frac{1}{2}\right)^{10}$.

5. The exponents of (4c) and $-\frac{3}{c}$ must be the same, that is 7. $\binom{14}{7}(4c)^7\left(\frac{3}{-c}\right)^7$ or $\binom{14}{7}4^7(-3)^7$.

SUMMARY:

1. What is a binomial expansion?
2. What is the statement of the Binomial theorem?
3. How do you find numbers for a row in Pascal's triangle?
4. How do you apply binomial expansions to solve probability problems?

LESSON: 48

AIM: How do you use data from an experiment to determine or verify probabilities?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Determine the frequency of outcomes in experiments.
2. Determine the probability of events based on experiments.
3. State the Law of Large Numbers.
4. Explain the use of samples to obtain estimates of probabilities.

VOCABULARY: statistics, descriptive statistics, statistical inference, frequency, law of large numbers, samples, histogram, bar chart

MOTIVATION:

Have each of your students roll a die twice. Write down the number of 1's, 2's, 3's, 4's, 5's, and 6's.

DEVELOPMENT:

1. The following column headings may be used to record the frequency of each tally: outcome, tallies, frequency. Using probability, what would you expect for the results of the motivation problem? How closely do the results of the die rolling correspond to these expectations? Explain to the class that **statistics** is a branch of mathematics which consists of the collection of data and the totality of methods used in collecting, processing, analyzing, and interpreting data. **Descriptive statistics** consists of summarizing and describing data. **Frequency distributions**, such as used for the motivation problem, present data in a compact form. The use of probability theory as the basis for the methods used to generalize from observed data is called **statistical inference.** One would expect that the more times that one rolled a die the closer one's result would be to the expected values

from probability theory. In fact, there is a mathematical law, the **Law of Large Numbers**, that states: **If the probability of an outcome X has a probability of P(X) and an experiment is performed many times, then the** $\frac{\textbf{number of times that X occurs}}{\textbf{total number of trials}}$ **approximately equals P(X).**

2. A population is a set of data consisting of all possible observations of a certain phenomenom. Eventhough a population is finite, such as the voters in the United States, it may not be practical to query every U.S. voter. Instead, a polltaker will work with a sample, that is a set of data containing part of the observations of voters. It is important that the sample be unbiased, in that different types of people from different parts of the country are included in a survey. For example, a sample of voters from only the Northeast would not be expected to accurately predict the outcome of a presidential election. What criteria sould be considered in selecting a sample of U.S. voters to accurately predict a presidential election?

3. What are examples of different types of statistical data? Different types are: (a) nominal - categories not to be used for mathematical calculations, such as marital status; (b) ordinal - categories which can be numerically compared, such as the hardness of metal; (c) interval - categories which can be subtracted but not multiplied or divided, such as temperature; (d) ratio - categories which form quotients, such as length and amounts of money.

APPLICATIONS:

* 1. Statistical data may be analyzed by distributions other than frequency distributions. Consider the following frequency distribution regarding the number of basketball players for each range of heights:

Heights of players (rounded to nearest inch)	Frequency
5'5" - 5'7"	6
5'8" - 5'10"	20
5'11" - 6'1"	28
6'2" - 6'4"	50
6'5" - 6'7"	34
6'8" - 6'10"	18
6'11" - 7'1"	4
Total	160

A percentage distribution table is obtained by dividing each class (frequency listing) by the total number of items grouped and then multiplying by 100. A cumulative distribution table is formed by adding the class distributions, starting at the top or bottom. Using the frequency distribution, form a percentage distribution and a cumulative distribution.

2. If you wanted to know what students in your school think about the math requirements, how would you choose a sample of 100 students in your school?

* 3. Data is often represented graphically. In a histogram, measurements are drawn on a horizontal scale with frequencies on a vertical scale. In a bar chart, there is no continuous horizontal scale. For the frequency distribution of application problem (1), draw a histogram and bar chart.

* 4. Draw parallel lines which are a distance apart which is equal to the length of a given needle. Throw the needle 100 times and determine the probability that the needle intersects 1 line. Express the probability in terms of π.

SOLUTIONS TO APPLICATION PROBLEMS:

1. Percentage distribution.

Heights	Percentage
5'5" - 5'7"	3.75
5'8" - 5'10"	12.50
5'11" - 6'1"	17.50
6'2" - 6'4"	31.25
6'5" - 6'7"	21.25
6'8" - 6'10"	11.25
6'11" - 7'1"	2.50
	100.00

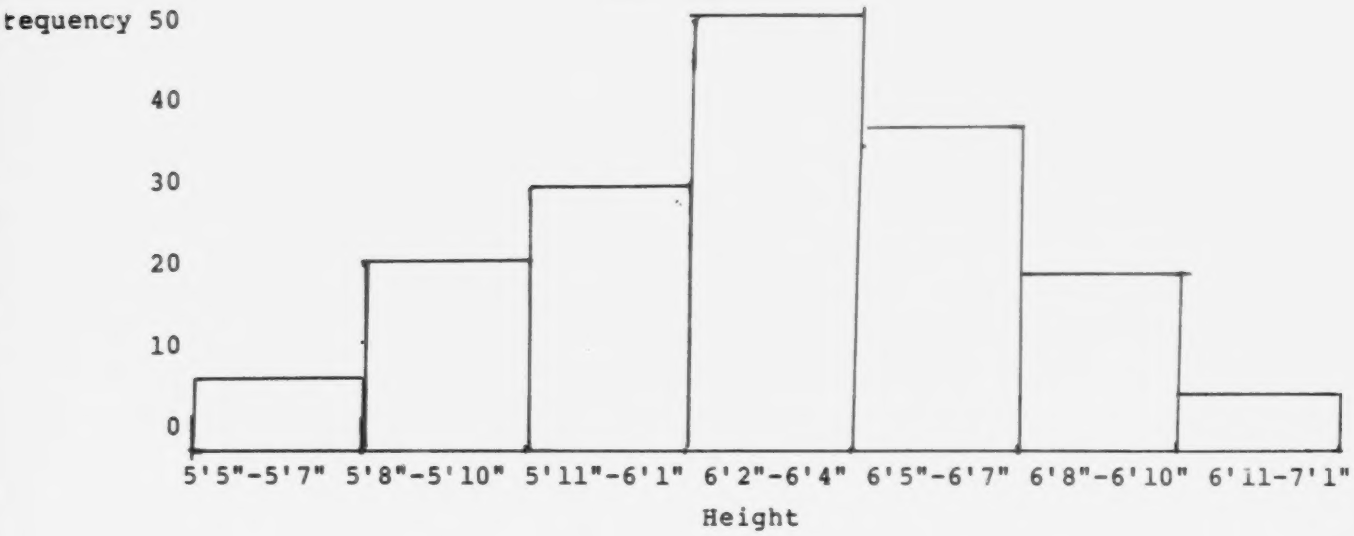
Cumulative distribution (less than).

Heights	Cumulative Frequency
Less than 5'5"	0
Less than 5'8"	6
Less than 5'11"	26
Less than 6'2"	54
Less than 6'5"	104
Less than 6'8"	138
Less than 6'11"	156
Less than 7'2"	160

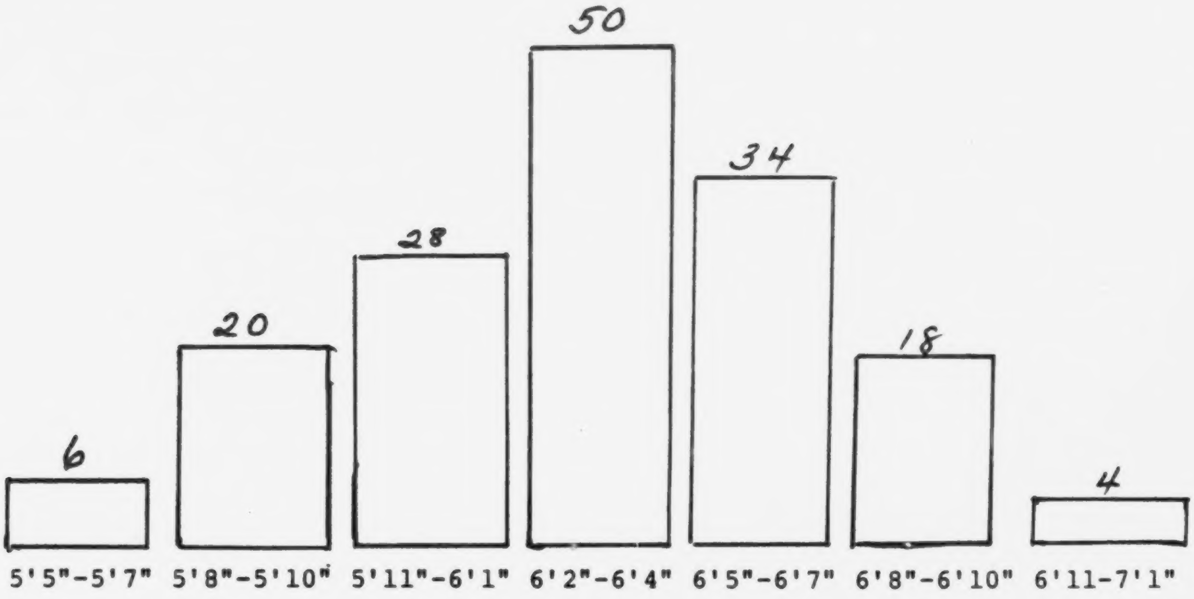
2. You might randomly choose 25 students from each grade (from regular and honor classes) so that the percentage of boys and girls and ethnic composition in the sample corresponds to the general student body.

3.

Histogram



Bar Chart



4. This is a famous probability problem, called Buffon's needle problem. The probability that the needle intersects one line is $\frac{2}{\pi}$.

SUMMARY:

1. What does statistics mean?
2. What is the Law of Large Numbers?
3. What is a population?
4. What is a sample?
5. What is descriptive statistics?
6. What is statistical inference?

LESSON: 49

AIM: How is the summation sign used to write a formula for the mean of a set of data?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Find the mean of a set of data.
2. Use summation notation to express sums.
3. Evaluate summations of linear or quadratic expressions.
4. Use the formula $\sum_{i=1}^{n} i=\frac{n(n+1)}{2}$ to evaluate sums.

VOCABULARY: arithmetic mean, measure of central tendency, summation notation, linear expressions, quadratic expressions

MOTIVATION:

If your grades this year are 85, 92, 91, 80, 94, 85, 91, 95, and 92, what grade do you need in your remaining course in order to have a 90 average?

DEVELOPMENT:

1. Elicit from the class that the word "average" is generally understood to be arithmetic mean. The **arithmetic mean, $\overline{X}$, of n numbers** (where n is a positive integer) equals: $\frac{\textbf{sum of numbers}}{n}$.

   For the motivation problem, let y = grade in the remaining course. $90 = \frac{85+92+91+80+94+85+91+95+80+y}{10}$

   $90 = \frac{805+y}{10}$ $\quad 900-805=y \quad y=95$

2. An arithmetic mean is a measure of central tendency, since it is one centrally located number. Since the arithmetic mean involves sums, there is an efficient symbol that was developed to indicate sums. In summation notation, the sum of $X_1+X_2+...+X_n$ is written with sigma

notation as $\sum_{i=1}^{n} X_i$. The sigma notation means the sum of X_i for i=1 to i=n, in increments of 1. The variable i is a dummy variable and the numbers 1 and n are the limits of summation. Consider the following examples:

(a) $\sum_{i=1}^{7} i = 1+2+3+4+5+6+7 = 28.$

(b) $\sum_{i=1}^{5} i^2 = 1^2+2^2+3^2+4^2+5^2 = 55.$

(c) $\sum_{i=4}^{8} 3i = 3\cdot4+3\cdot5+3\cdot6+3\cdot7+3\cdot8 = 3(4+5+6+7+8) = 90.$

(d) $\sum_{i=1}^{4} (X_i+y_i) = (X_1+y_1)+(X_2+y_2)+(X_3+y_3)+(X_4+y_4),$

which can be rewritten as $(X_1+X_2+X_3+X_4)+(y_1+y_2+y_3+y_4)$ or $\sum_{i=1}^{4} X_i + \sum_{i=1}^{4} y_i$.

3. Generalizing 2(a), the following formula can be used to find the sum of the first n positive integers:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

As an example, $\sum_{i=1}^{100} i = \frac{100(100+1)}{2} = 5050$

From factoring out the 3 in problem (2c), it can be seen that the following formula holds:

$\sum_{i=1}^{n} Ki = K\sum_{i=1}^{n} i$ for constant K. For example, $\sum_{i=1}^{50} 7i = 7\sum_{i=1}^{50}$ $= \frac{7\cdot50(50+1)}{2} = 8925$

Generalizing from (2d), $\sum_{i=1}^{n}(X_i+y_i) = \sum_{i=1}^{n} X_i+\sum_{i=1}^{n} y_i$

4. Using the summation notation, what is the formula for the arithmetic mean?

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

APPLICATIONS:

1. Write the Binomial theorem using summation notation.
2. Given: $X_1=12,\ X_2=4,\ X_3=7,\ X_4=9,\ X_5=17,\ X_6=5$.

   What is $\frac{1}{6}\sum_{i=1}^{6}\left|\overline{X}-X_i\right|$?
3. Find $\sum_{i=61}^{80} i$ (hint: write as a difference of sums).

* 4. Use summation notation to find the sum of the first 100 even integers.

* 5. Use summation notation to find the sum of the first 100 odd integers.

SOLUTIONS TO APPLICATION PROBLEMS:

1. Binomial theorem: $(x+y)^n = {}_nC_0x^n + {}_nC_1x^{n-1}y^1 + {}_nC_2x^{n-2}y^2 + \ldots + {}_nC_ny^n$ or $(x+y)^n = \sum_{r=0}^{n} {}_nC_rx^{n-r}y^r$.

   Notice that the sum of the exponents, n-r+r, is n and that the exponent of y is the same as the dummy variable, r.

2. $\overline{X} = \frac{1}{6}\sum_{i=1}^{6} X_i = \frac{1}{6}(12+4+7+9+17+5) = 9.$

   $\frac{1}{6}\overset{6}{\underset{i=1}{}}\ \overline{X}-X_i = \frac{1}{6}(|12-9|+|4-9|+|7-9|+|9-9|+|17-9|+|5-9|) =$

   $\frac{1}{6}(3+5+2+0+8+4) = \frac{22}{6}$ or $3.\overline{6}$.

   In a future lesson, the mean absolute deviation (a measure of dispersion, which tells how scattered the measures are) will be defined as $\frac{1}{n}\sum_{i=1}^{n}\left|\overline{X}-X_i\right|$.

3. $\sum_{i=61}^{80} i = \sum_{i=1}^{80} i - \sum_{i=1}^{60} i = \frac{80(80+1)}{2} - \frac{60(60+1)}{2} = 3240-1830$

   $= 1410.$

4. There are 50 even integers from 1 to 100. Each even integer can be expressed as 2i, where i = 1,2,...50.

$$\sum_{i=1}^{50} 2i = 2\sum_{i=1}^{50} i = \frac{2\cdot 50(50+1)}{2} = 2550.$$

5. There are 50 odd integers from 1 to 100. Each odd integer can be expressed as 2i-1, where i = 1,2,...50.

$$\sum_{i=1}^{50} (2i-1) = 2\sum_{i=1}^{50} i - \sum_{i=1}^{50} 1 = 2550-50 = 2500.$$

Note that $\sum_{i=1}^{100} i = 5050$ = sum of even integers plus sum of odd integers = 2550+2500.

SUMMARY:

1. What is an arithmetic mean?
2. What is a measure of central tendency?
3. What are applications of sigma notation?

LESSON: 50

AIM: How does one use frequency tables to organize data and find the mean?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Set up a frequency table of data.
2. Use a frequency table to draw a histogram.
3. Determine the mean from a frequency table and histogram.

VOCABULARY: frequency table, histogram

MOTIVATION:

What is the arithmetic mean of the following data obtained from a survey of 34 college students about the number of courses they were taking this term: 5,4,6,4,4,5,7,6,5,4, 4,5,7,4,5,6,4,4,5,7,7,4,5,4,6,4,4,7,6,6,5,5,5,6?

DEVELOPMENT:

1. What methods can be used to find the arithmetic mean of the data from the survey? Elicit: The summation formula, $\frac{1}{34}\sum_{i=1}^{34} X_i$, can be used. But since each of the four values occurs frequently, the following frequency table might be helpful to organize the data and find the arithmetic mean:

Number of Courses	Frequency
4	12
5	10
6	7
7	5

The sum of the data equals $4\cdot 12+5\cdot 10+6\cdot 7+7\cdot 5$, or 175, which represents the sum of each number times its frequency. The mean = $\frac{175}{34}$ or 5.15 to the nearest hundredth. To generalize the method of using frequencies to find the arithmetic mean, the following table may be used. (Where X_i represents each number in the data, f_i represents the frequency, and $X_i f_i$ represents the product of each number times its frequency):

index	X_i	f_i	$X_i f_i$
1	4	12	48
2	5	10	50
3	6	7	42
4	7	5	35

The sum is $\sum_{i=1}^{4} X_i f_i$. The mean is $\dfrac{\sum_{i=1}^{4} X_i f_i}{\sum_{i=1}^{4} f_i}$

The general formula for **finding the arithmetic mean, $\overline{X}$, of n items,** with j distinct values, and where f_i is the frequency of each value X_i is:

$$\overline{X} = \frac{\sum_{i=1}^{j} X_i f_i}{\sum_{i=1}^{j} f_i}$$

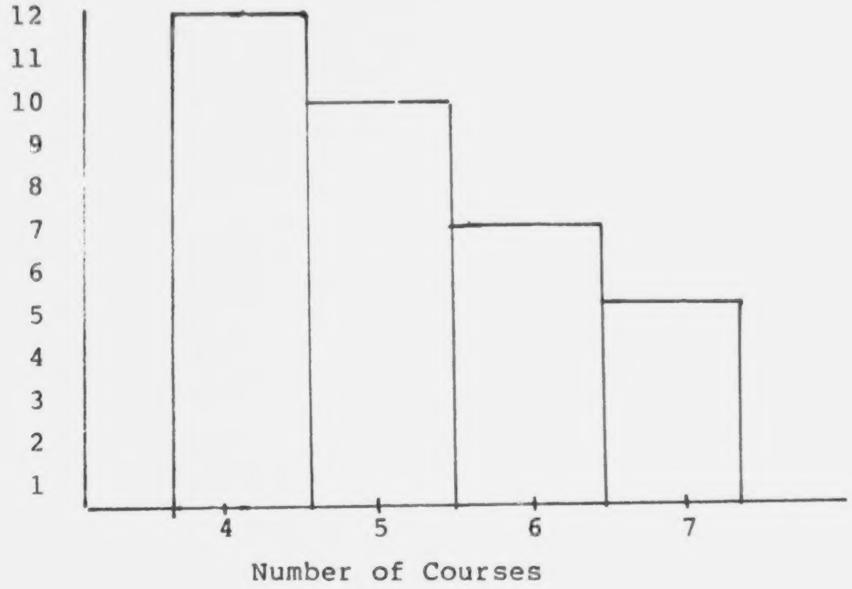
As another example, find the mean from the following frequency table of monthly precipitation.

index	Precipitation- to nearest inch, X_i	Number of months, f_i	$X_i f_i$
1	2	3	6
2	3	2	6
3	4	1	4
4	5	4	20
5	6	2	12

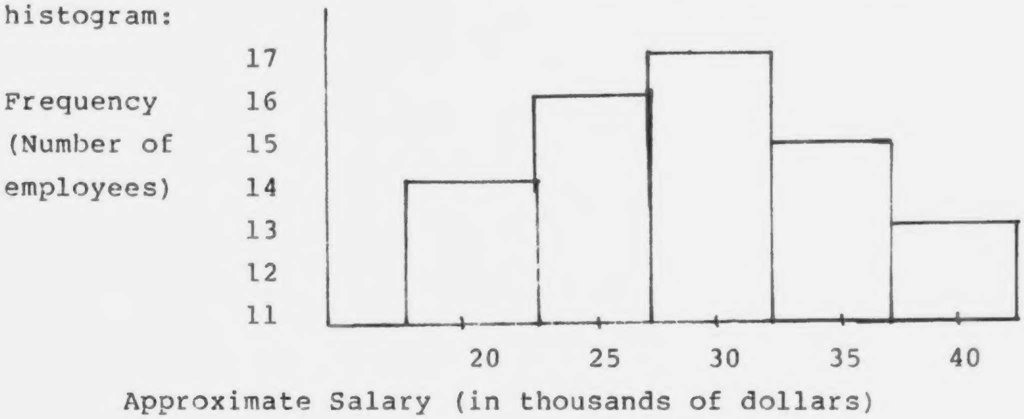
$$\overline{X} = \frac{6+6+4+20+12}{3+2+1+4+2} = \frac{48}{12} = 4 \text{ inches.}$$

2. Explain to the class that to provide a graphical representation of data, a **histogram** may be drawn. In a histogram the measurements are drawn on a horizontal scale with frequencies on a vertical scale. The height of each rectangle represents the frequency of that particular value in the data. The following histogram

was drawn using the data from the motivation problem:



3. Find the mean of the data represented by the following histogram:



$$\overline{X} = \frac{14\cdot20 + 16\cdot25 + 17\cdot30 + 15\cdot35 + 13\cdot40}{14 + 16 + 17 + 15 + 13} = \frac{2235}{75}$$

= \$29,800

APPLICATIONS:

1. Take a survey of students in your class. You might ask students their ages to the nearest month. Organize the data into a frequency table and histogram.

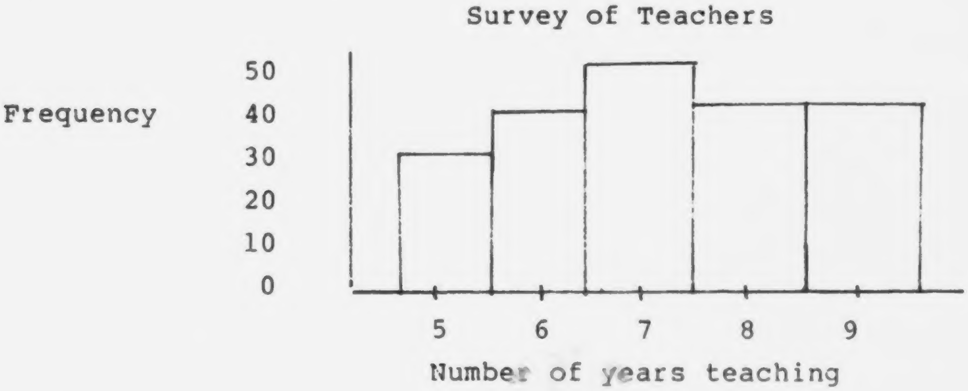
2. The following frequency table represents the grades (given as letters) of a class:

index	grade	f_i
1	A	7
2	B	11
3	C	9
4	D	3

This is an example of a frequency table with interval data, that is each letter represents a numerical interval. Assume the following intervals:
A - 90 to 100; B - 80 to 89; C - 70 to 79; D - 65 to 69.

In order to find an approximate value of $\bar{X}$, use the midpoint of each interval. This midpoint is called the class mark. Find an approximate value of the mean.

* 3. Percentiles may be determined from a histogram. Find the 60th percentile from the following histogram:



SOLUTIONS TO APPLICATION PROBLEMS:

2. Rewrite the frequency table as:

index	grade	class mark, X_i	f_i	$X_i f_i$
1	A	95	7	665
2	B	84.5	11	929.5
3	C	74.5	9	670.5
4	D	67	3	201

$$\text{(Approximate) } \bar{X} = \frac{665+929.5+670.5+201}{30} = \frac{2466}{30} = 82.2$$

3. The 60th percentile represents the number of years experience at or below which the lowest 60% of the teachers are found. Since the sum of 30 teachers (with 5 years experience) plus 40 teachers (with 6 years experience) plus 50 teachers (with 7 years experience) is 120, the 60th percentile is 7 (60% times 200 is 120).

SUMMARY:

1. What should the columns of a frequency table be in order to find the mean of a set of data?
2. Using the frequency of each value in the data, what is the formula for the mean?
3. What is a histogram?

LESSON: 51

AIM: How can a single number, or statistic, be determined that represents all the numbers in a set of data?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define measures of central tendency.
2. Find the median and mode of a set of data from a list, frequency table, or histogram.

VOCABULARY: measures of central tendency, median, mode

MOTIVATION:

In town A, there are 99 people with yearly salaries of $10,000 and one person with a yearly salary of $5,000,000. In town B, there are 100 people with yearly salaries of $50,000. What are the arithmetic means of the salaries in each town?

DEVELOPMENT:

1. Elicit that the arithmetic means of the salaries are $59,900 for town A and $50,000 for town B. For this motivation problem, the arithmetic mean does not seem to represent the numbers in the data since the arithmetic mean is higher for town A than town B largely because of only one person in town A. Explain to the class that there is a need for another single number, or statistic, to represent all the numbers in a set of data. These single numbers are called measures of central tendency. Examples of measures of central tendency are: (a)**arithmetic mean;** (b) **median;** and (c) **mode.** If the data is in increasing or decreasing order, **the median is defined as the middle item or the mean of the two middle terms.** For the motivation problem, the median for town A is $10,000 and for town B is $50,000. **The mode is defined as the value with the greatest frequency.** The mode for town A is $10,000 and for town B it is $50,000.

For this motivation problem, the median and mode both give "truer" pictures of the "average" salary for town A than does the arithmetic mean.

2. Consider the following list of car sales in a 6 month span:

Months	Number of cars
Jan.	4
Feb.	9
March	7
April	4
May	6
June	10

(a) What is the median sale? Arrange the numbers in ascending (or descending) order: 4,4,6,7,9,10. Since there are an even number (6) of sales, the median is the arithmetic mean of the middle numbers, or $\frac{6+7}{2} = 6.5$.

(b) What is the mode? The most frequently occurring value is 4.

(c) What is the arithmetic mean? $\overline{X} = \frac{4+9+7+4+6+10}{6} = 6\frac{2}{3}$

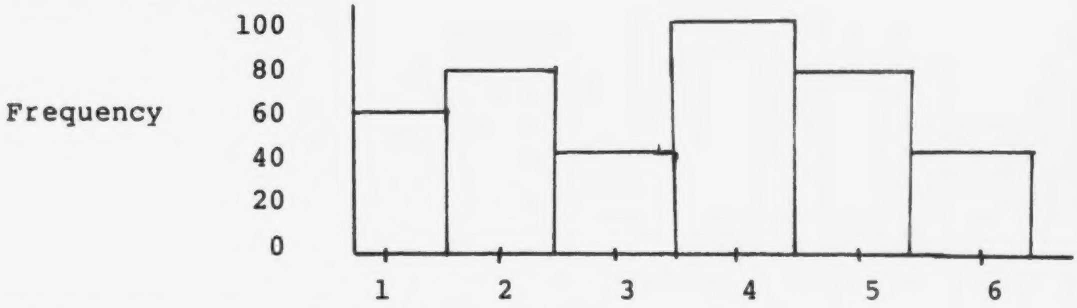
3. Consider the following frequency table of numbers of books sold to each customer in one week.

Number of books	Frequency
1	150
2	162
3	93
4	104
5	75
6	42
7	31

(a) What is the median value of the number of books sold per customer? First find the total number of books sold, which is 657. The middle value for frequency is 329. The number of books corresponding to 329 sales is 3, since 150+162 is 312 and 150+162+93 is 405. The median number of books sold is 3.

(b) What is the mode of the number of books sold per customer? The mode is 2, since it has the greatest frequency.

4. Find the median and mode of the data values from the following histogram:



Assume the data values are 1,2,3,4,5, and 6.

(a) To find the median, find the total frequency: 60+80+40+100+80+40 = 400. The middle value for frequency is 200.5. The data value corresponding to 200.5 is 4, since 60+80+40 is 180 and 60+80+40+100 is 280. (b) The mode, the data value occurring with the greatest frequency is 4.

Note: If there are n items and n is odd, the median is the value of the $\frac{n+1}{2}$th largest item. If n is even, the median is the arithmetic mean of the values of the $\frac{n}{2}$th and $\left(\frac{n}{2}+1\right)$th largest items.

APPLICATIONS:

1. The following test results were obtained on an exam: 75,90,55,83,72,91,75,88,60,94,83,68.
   a) What are the mean, median, and mode?
   b) If the test scores are all raised by 6 points, what are the mean, median, and mode?
   c) How do the mean, median, and mode for the new test scores compare with the original test scores?

2. What is the median value for the following set of data: $5x-2$, $6x$, $2x+5$, x^2, $9x-x^3$, $4x+3$, and $3x+7$ for $x=3$?

* 3. The harmonic mean of n numbers $a_1, a_2, a_3, \ldots a_n$ is defined as $\frac{n}{\sum \frac{1}{a}}$. Use the harmonic mean to determine the

average speed of a car which drives 20 miles on a highway at 60 miles per hour and then 20 miles at 20 miles per hour.

SOLUTIONS TO APPLICATION PROBLEMS:

1. a) Mean = $77\frac{5}{6}$. Median = 79. Modes are 75 and 83 (the scores are bimodal).
   b) Mean = $83\frac{5}{6}$. Median = 85. Modes are 81 and 89.
   c) The new mean, median, and mode are all 6 points higher than the original test scores.

2. The numbers are 13,18,11,9,0,15, and 16. In ascending order the numbers are 0,9,11,13,15,16, and 18. The median is 13.

3. Harmonic mean = $\frac{2}{\frac{1}{60} + \frac{1}{20}} = \frac{2}{\frac{1}{60} + \frac{3}{60}} = \frac{2}{\frac{4}{60}}$ = 30 mph.

SUMMARY:

1. What are measures of central tendency?
2. What is the median?
3. What is the mode?

LESSON: 52

AIM: What are the statistics that measure how much data is scattered?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define measures of dispersion.
2. Find the range of a set of data.
3. Find the mean absolute deviation of a set of data.
4. Determine variance and standard deviation as measures of dispersion.

VOCABULARY: measures of dispersion, range, mean absolute deviation, variance, and standard deviation

MOTIVATION:

Assume the test scores in a course for Pedro are 80,90,85,88,82,85, and 85, while for Julie the test scores are 85,100,100,100,70,70, and 70. What are the median and mean for the test scores of Pedro and Julie?

DEVELOPMENT:

1. Even though the median and mean for both Pedro's and Julie's test scores are 85, the test scores are obviously different. There is a need for another type of measure besides the measures of central tendency; a statistic which measures how scattered the data is. This statistic, which measures the dispersion of data, is called a measure of dispersion. The simplest measure of dispersion is the range, which is defined as the largest minus the smallest value. The range for Pedro's grades is 10 and for Julie's grades is 30. The range is of limited value, though, because it only involves the two extreme values of the data. A measure of dispersion which gives a better idea of the amount of scattering of data would involve every value of the data.

2. To which value of the data might each of the values of the data be compared to? Each value in the data might be compared to a "central" value, the arithmetic mean. Let $\overline{X}$ = arithmetic mean, X_i = ith value in the data, n = total number of values of the data. The mean absolute deviation is defined as $\frac{1}{n}\sum_{i=1}^{n}|X_i-\overline{X}|$.

   Calculate the mean absolute deviations for Pedro and Julie as follows:

Pedro		Julie	
X_i	$\|X_i-\overline{X}\|$	X_i	$\|X_i-\overline{X}\|$
80	5	85	0
90	5	100	15
85	0	100	15
88	3	100	15
82	3	70	15
85	0	70	15
85	0	70	15
	$\sum_{i=1}^{7}\|X_i-\overline{X}\|=16$		$\sum_{i=1}^{7}\|X_i-\overline{X}\|=90$

   The mean absolute deviation for Pedro is $\frac{16}{7}$ or 2.3 (to the nearest tenth), while the mean absolute deviation for Julie is $\frac{90}{7}$ or 12.9. The mean absolute deviation for Julie is higher than for Pedro, which is what one would expect, since Julie's scores are more scattered than Pedro's. Since the sum, $\sum_{i=1}^{n} X_i-\overline{X}$ is always 0, the absolute value of $X_i-\overline{X}$ is taken.

3. Instead of taking the absolute value of $X_i-\overline{X}$, what other operation might one perform on $X_i-\overline{X}$ to make it always positive? One might take the square of $(X_i-\overline{X})$.

   Variance, s^2, is defined as: $s^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\overline{X})^2$.

Variance is the average value of the squares of the deviations from the mean. The variances for Pedro's scores and Julie's scores are:

Pedro			Julie		
x_i	$\lvert x_i-\bar{X}\rvert$	$(x_i-\bar{X})^2$	x_i	$\lvert x_i-\bar{X}\rvert$	$(x_i-\bar{X})^2$
80	5	25	85	0	0
90	5	25	100	15	225
85	0	0	100	15	225
88	3	9	100	15	225
82	3	9	60	15	225
85	0	0	60	15	225
85	0	0	60	15	225
		$\sum_{i=1}^{7}(X_i-\bar{X})^2=68$			$\sum_{i=1}^{7}(X_i-\bar{X})^2=1350$

$s^2 = \frac{68}{7}$ or 9.7 $s^2 = \frac{1350}{7}$ or 192.9

4. Since the variance may be a large number, another statistic, the standard deviation, S, is more commonly used. The standard deviation is the square root of the variance, that is,

$$S = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{X})^2}.$$

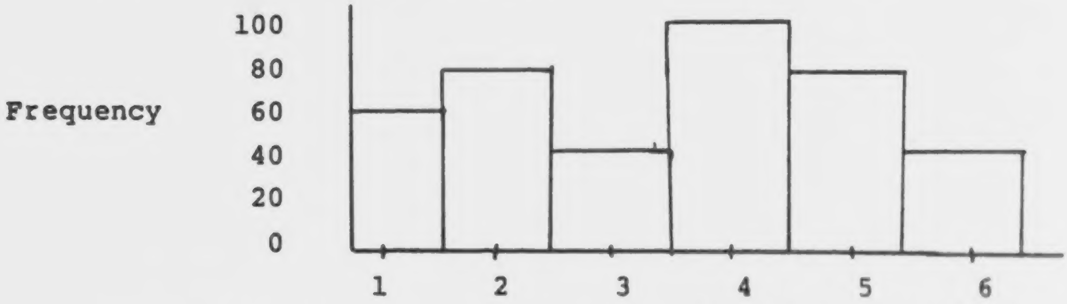
For Pedro, $s \approx \sqrt{9.7} \approx 3.1$ For Julie, $s \approx \sqrt{192.9} \approx 13.9$

APPLICATIONS:

1. Find the variance and standard deviation for the data in the following frequency table:

Number of books sold to each customer	Frequency
1	150
2	162
3	93
4	104
5	75
6	42
7	31

2. Find the variance and standard deviation for the data from the following histogram:



3. Assume that the mean on a test was 80 and the standard deviation was 9. The teacher decides that all scores within one standard deviation of the mean will be assigned a grade of B. What scores will be assigned a grade of B?

4. The average monthly temperature (in Fahrenheit) is shown below for cities A and B:

	Jan	Feb	March	April	May	June	July	Aug	Sept	Oct	Nov	Dec
City A	30	28	36	42	58	67	75	72	54	40	37	31
City B	27	31	39	45	54	71	77	76	61	38	33	29

Which city has the more variable climate?

SOLUTIONS TO APPLICATION PROBLEMS:

1. $\overline{X} = (150+162+93+104+75+42+31)/7 = 657/7$

$\overline{X} = 93.9$ (to nearest tenth).

X_i	$\lvert X_i-\overline{X}\rvert$	$(X_i-\overline{X})^2$
150	56.1	3147.2
162	68.1	4637.6
93	.9	.8
104	10.1	102.0
75	18.9	357.2
42	51.9	2693.6
31	62.9	3956.4

$$\sum_{i=1}^{7}(X_i-\overline{X})^2 = 14894.8$$

$$\text{Variance} = s^2 \approx \frac{1}{7}(14894.8) \quad 2127.8$$

$$s \approx \sqrt{2127.8} \approx 46.1$$

2. $\overline{X} = (60+80+40+100+80+40)/6 = 400/6$

$\overline{X} = 66.7$

X_i	$\lvert X_i-\overline{X}\rvert$	$(X_i-\overline{X})^2$
60	6.7	44.9
80	13.3	176.9
40	26.7	712.9
100	33.3	1108.9
80	13.3	176.9
40	26.7	712.9

$$\sum_{i=1}^{6}(X_i-\overline{X})^2=2933.4$$

$$\text{Variance} = s^2 \approx \frac{1}{6}(2933.4) \approx 488.9$$

$$s \approx \sqrt{488.9} \approx 22.1$$

3. One standard deviation equals 9, so that scores from 71 to 89 will receive a B.

4. $\overline{X} = 570/12$

$\overline{X} = 47.5$

City A

X_i	$\lvert X_i-\overline{X}\rvert$	$(X_i-\overline{X})^2$
30	17.5	306.3
28	19.5	380.3
36	11.5	132.3
42	5.5	30.3
58	10.5	110.3
67	19.5	380.3
75	27.5	756.3
72	24.5	600.3
54	6.5	42.3
40	7.5	56.3
37	10.5	110.3
31	16.5	272.3

$$\sum_{i=1}^{12}(X_i-\overline{X})^2=3177.6$$

$$s^2 \quad \frac{1}{12}\cdot(3177.6) \approx 264.8 \quad ; \quad s \approx \sqrt{264.8} \approx 16.3$$

$\bar{X} = 581/12$

$\bar{X} = 48.4$

City B

X_i	$\lvert X_i-\bar{X}\rvert$	$(X_i-\bar{X})^2$
27	21.4	458.0
31	17.4	302.8
39	9.4	88.4
45	3.4	11.6
54	5.6	31.4
71	22.6	510.8
77	28.6	818.0
76	27.6	761.8
61	12.6	158.8
38	10.4	108.2
33	15.4	237.2
29	19.4	376.4

$$\sum_{i=1}^{12} (X_i - \bar{X})^2 = 3863.4$$

$$s^2 \approx \frac{1}{12} \cdot (3863.4) \approx 322.0 \quad ; \quad s \approx \sqrt{322.0} \approx 17.9$$

Therefore, City B has a more variable climate.

SUMMARY:

1. What are measures of dispersion?
2. What is the range?
3. What is the mean absolute deviation?
4. What is the variance?
5. What is the standard deviation?

LESSON: 53

AIM: How can one determine the probabilities of the results of an experiment which has a normal distribution?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Convert from the original values of an experiment which has a normal distribution to Z scores.
2. Use a table of values to determine the percent of a given population, with a normal distribution, which fall within a certain range of values.

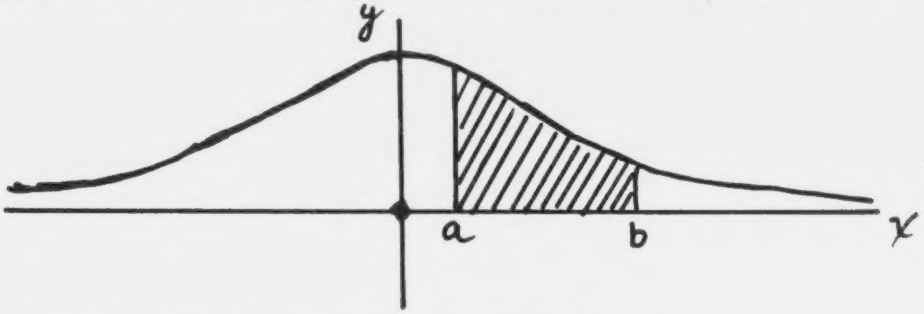
VOCABULARY: random variable, normal distribution, "bell shaped" curve, Z score

MOTIVATION:

Assume that the arithmetic mean of a standardized test, the math portion of the PSAT, is 48 and the standard deviation is 10. What is the probability that a student scores from 48 to 60, inclusive, on the math portion of the PSAT?

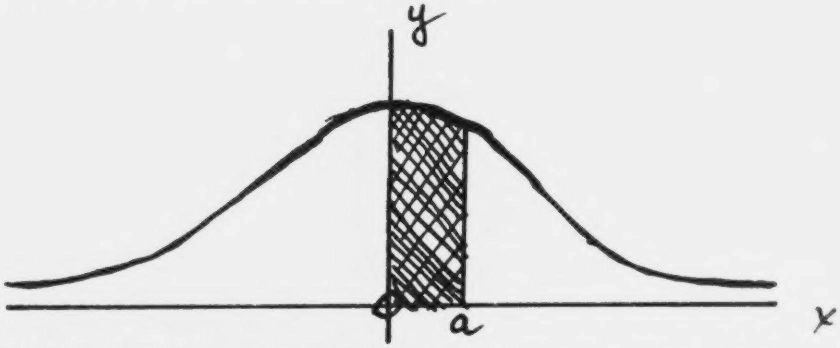
DEVELOPMENT:

1. Explain to the class that standardized test scores and biological traits, such as height and weight, approximate a normal distribution, whose graph is called a "bell shaped" curve. The graph of a normal distribution is:



Normal distributions have the property that the area under the curve between values a and b gives the probability that the random variable (the quantity which takes on different values due to chance) will assume a value on the interval from a to b. Therefore, the total area above the x-axis and below the curve is 1. The curve is symmetric about a vertical line through the mean, $\overline{X}$.

2. A normal distribution table, which will be used to determine probabilities, can be found in any sequential math textbook. To use the table, the scale of the x-axis must be changed so that the mean is 0 and the standard deviation is 1. The new axis is called the z-axis. The table gives the probability that a value of z will be between the mean, 0, and value a.



The following formula can be used to convert from the original values of the problem, x values, to z scores: $Z = \frac{X-\overline{X}}{s}$ ($\overline{X}$ = mean, s = standard deviation).

Note that if X equals $\overline{X}$, then Z equals 0. If $X = \overline{X}+s$, that is X is within a standard deviation of the mean, then $Z = \frac{\overline{X}+s-\overline{X}}{s} = 1$. As an example of an application of the formula for Z, consider the motivation problem: $\overline{X} = 48$, $s = 10$, and $X = 60$. $Z = \frac{60-48}{10} = 1.2$.

In the normal distribution table, the number in the column, $P(0 \le z \le a)$, next to 1.2 is 0.385. Therefore, the probability that a person has a score ranging from the mean value of 48 to 60 is 0.385.

3. Using the given information from the motivation problem, what is the probability that a person scores:

a) from 20, the lowest score, to 60;

b) above 60?

In these two examples, one is asked to find the probability of a score that is not in the range from a z score of 0 to a. Since the total probability of all scores equals 1, the probability of a z score to the left of the y-axis equals .5 and the probability of a z score to the right of the y-axis also equals .5. Thus, for part (a), the probability of a score from 20 to 60 equals the probability of a score from 20 to 48, which is .5, plus the probability of a score from 48 to 60, which is .385. The sum is .885. For part (b), the probability of a score above 60 equals 1-.885, or .115. Note that since we are assuming that standardized test scores approximately follow a normal distribution, all probability results are approximate.

APPLICATIONS:

1. Assume that the ages of senior high school students approximates a normal distribution, with mean 17 years and standard deviation .5 year. Approximately what percent of the students are: (a) 17.5 years old or older; (b) 16 years old or younger?
2. Assume that the amount of cosmic radiation that a person is exposed to on a flight from New York City to Paris is a random variable which has a normal distribution. The mean of the normal distribution is 5.4 mrem and the standard deviation is .3 mrem. What is the approximate probability that a person on a flight from New York City to Paris is exposed to 6 or more mrem of cosmic radiation?
3. Assume that the scores on an exam were normally distributed, with a mean of 56 and a standard deviation of 8. If the top 20% of the scores were assigned a grade of A, what score (to the nearest integer) was needed to get a grade of A?
4. Assume that the length of time that a certain television lasts is normally distributed. The mean of the length of time is 6 years, with a standard deviation of 1 year. The company replaces failed televisions if they are within the guarantee period. Approximately how long should the guarantee period be if the company does not want to replace more than 30% of the televisions?

SOLUTIONS TO APPLICATION PROBLEMS:

1. a) $Z = \frac{\bar{X}-X}{s}$ $Z = \frac{17.5-17}{.5} = 1.$ $.5 - .341 = .159$

   b) $Z = \frac{16-17}{.5} = -2.$ Look up Z = 2 in the table.

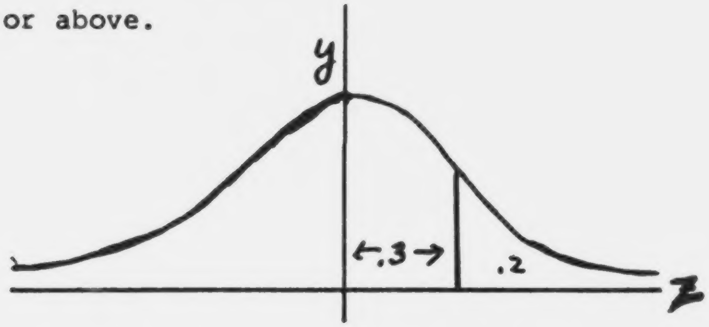
   $.5 - .477 = .023$

2. $Z = \frac{6-5.4}{.3} = 2.$ $.5 - .477 = .023$

3. The Z score which is closest to having a probability of .3(.5-.3=.2) is .8.

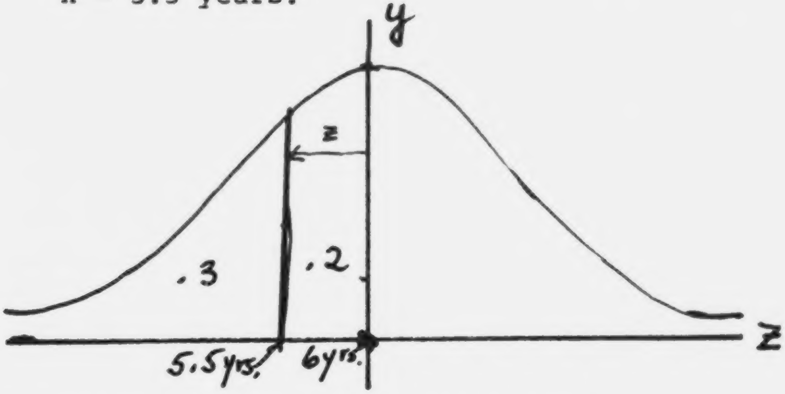
   $Z = .8 = \frac{X-56}{8}.$ $6.4 = X-56.$ $X = 62.4$

   "A" grade of 62 or above.



4. The Z score which is closest to having a probability of .2(.5-.3=.2) is .5.

   $Z = .5 = \frac{6-X}{1}.$ $X = 5.5$ years.



SUMMARY:

1. What are the properties of a normal distribution?

2. How are X values converted to Z scores?

3. How is the normal distribution table used to determine probabilities?

LESSON: 54

AIM: How do we compose transformations?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Draw accurate diagrams to illustrate the following theorems:
   a. If ℓ and m are 2 parallel lines, then R_m (R_ℓ) is a translation where the length of the translation is double the distance between lines ℓ and m and where the direction of the translation is perpendicular to the parallel lines.
   b. If ℓ and m are 2 intersecting lines, then R_m (R_ℓ) is a rotation about the point of intersection where the measure of the angle of rotation is twice the measure of the angle formed by lines ℓ and m (from ℓ to m).
2. Find image points of the composite of two translations, two rotations, or two dilations.

MOTIVATION:

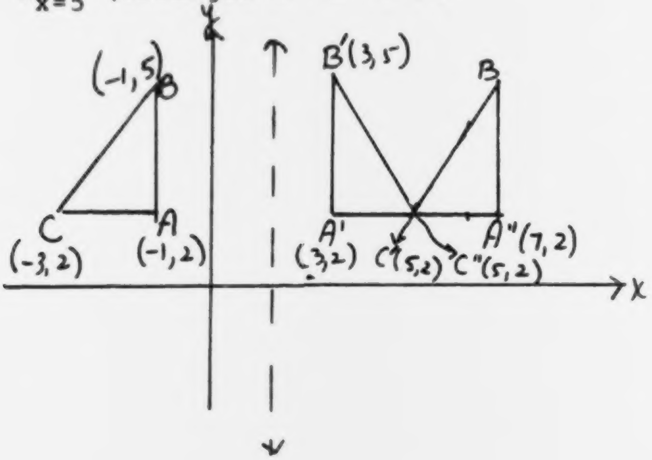
1. Graph triangle ABC with vertices A(-1, 2), B(-1, 5) and C(-3, 2).
2. Draw, on the same graph, the lines whose equations are $x = 1$ and $x = 5$.
   What geometric transformation is suggested by these graphs?

DEVELOPMENT:

1. The students should respond with one or two line reflections. A discussion on commutativity may occur naturally here if two students suggest doing composite reflections in opposite order.
   The teacher can facilitate that discussion by drawing and labelling a separate diagram.
2. Elicit: In the case of $R_{x=1}$ ($R_{x=5}$),
   A" has coordinates (-9, 2)
   B" has coordinates (-9, 5)
   C" has coordinates (-11, 2)

DEVELOPMENT:

3. Graph triangle A' B' C' where triangle A' B' C' = $R_{x=1}$ (triangle ABC).
4. Graph triangle A" B" C" where triangle A" B" C" = $R_{x=5}$ (triangle A' B' C').

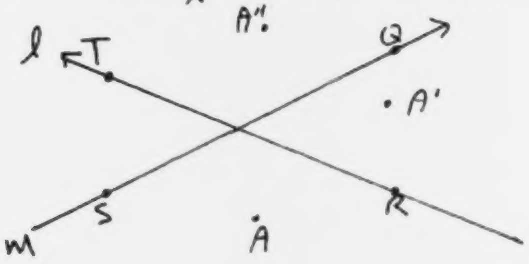


What single transformation is equivalent to $R_{x=5}$ $(R_{x=1})$? Students can visualize a translation.

How does the distance between the parallel lines compare with the distance of the translation?

THEOREM: R_m (R_ℓ) where ℓ and m are parallel lines is a translation whose distance is twice the distance between ℓ and m in the direction perpendicular to those lines.

5. Consider lines ℓ and m in the following diagram:



Find $R_\ell(A) = A'$

Then find $R_m\left[R_\ell(A)\right] = A''$

Measure $\measuredangle$ AOA" and Measure $\measuredangle$ RPQ.

What single transformation appears to be equivalent to $R_m(R_\ell)$? How can we determine the nature of the angle of rotation?

Note that $m\measuredangle AOA'' = 2\cdot m\measuredangle RPQ$.

State the following theorem:

If ℓ and m are two intersecting lines ($\ell \cap m = P$) than $R_m(R_\ell)$ is a rotation about the point of intersection where the measure of the angle of rotation is twice the measure of the angle formed by lines ℓ and m (from ℓ to m.)

APPLICATIONS:

1. Find the equations of 2 lines (ℓ and m) such that $R_m\left[R_\ell(D)\right] = D''$ if the coordinates of D and D' are (1, 0) and (7,0) respectively.
2. The coordinates of triangle ABC's vertices are A(2, -1), B(3, -1) and C(1, -2).
   - (a) Graph triangle ABC and the lines $x = -4$, and $y = 5$.
   - (b) State the coordinates of the vertices of triangle A" B" C", the image of triangle ABC under $R_{x\text{ - axis}}(R_{y=5})$.
   - (c) What single geometric transformation is equivalent to the composite $R_{x\text{ - axis}}(R_{y=5})$?
   - (d) State the coordinates of the vertices of triangle A" B" C", the image of triangle ABC under $R_{y\text{ - axis}}(R_{x=-4})$.
   - (e) Describe a single geometric transformation equivalent to the composite $R_{y\text{ - axis}}(R_{x=-4})$.
   - (f) State the coordinates of the vertices of triangle A" B" C", the image of triangle ABC under $R_{y\text{ - axis}}(R_x)$.
   - (g) What single transformation is equivalent to $R_{y\text{ - axis}}(R_x)$?

(h) State the coordinates of the vertices of triangle A" B" C", the image of triangle ABC under

(i) $T_{2,1} \cdot T_{-3,5}$

(ii) $D_2 \cdot D_3$

(iii) $\text{Rot}_{90^\circ} \cdot \text{Rot}_{180^\circ}$

What single transformation is equivalent to each composite?

SOLUTIONS TO APPLICATION PROBLEMS:

1. Different pairs are acceptable. E.g., $x = 3$ and $x = 6$, or $x = 2$ and $x = 5$, or $x = 1\frac{1}{2}$ and $x = 4\frac{1}{2}$, etc.
   What common feature describes each answer?
   Each pair of lines is separated by 3 units. That is one-half the distance between D and D".

Thm: Every translation can be expressed as a composite of reflection in parallel lines where the distance between the lines is one-half the distance of the translation and the lines are perpendicular to the direction of the translation.

2. See attached graph paper.

SUMMARY:

What effects do composing transformations have, where the transformations are the same type?

$R_\ell \circ R_m = T$ where $\ell \parallel m$

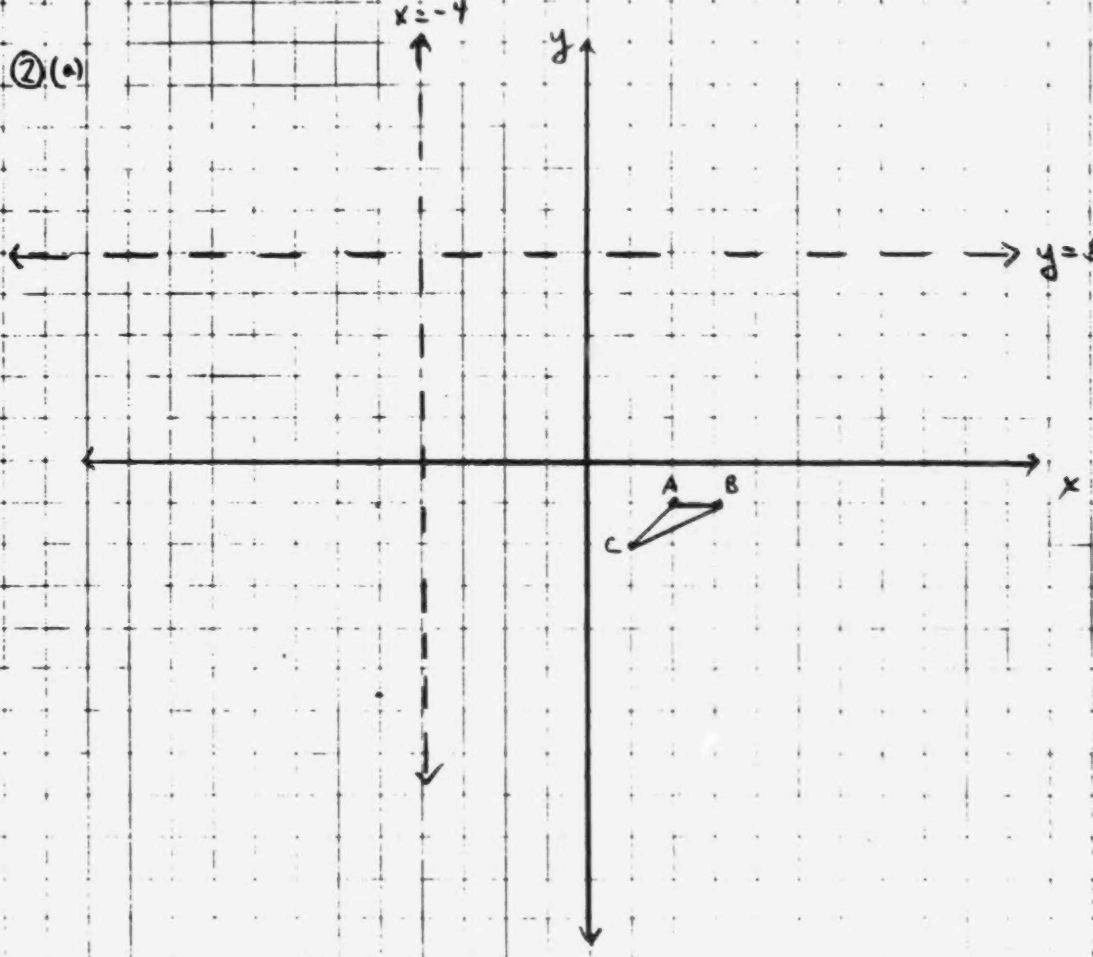
$R_\ell \circ R_m = \text{Rot}$ where $\ell \cap m = P$

$T_{(a,b)} \circ T_{(c,d)} = T_{(a+c,\ b+d)}$

$\text{Rot}_A \cdot \text{Rot}_B = \text{Rot}_{A+B}$

$D_a \cdot D_b = D_{ab}$

②(a)



(b) A" (2,-11)
B" (3,-11)
C" (1,-12)

(c) $T_{0,-10}$

(d) A" (10,-1)
B" (11,-1)
C" (9,-2)

(e) $T_{8,0}$

(f) A" (-2,1)
B" (-3,1)
C" (-1,2)

(g) $Rot_{0,180°}$

(h) (i) A" (1,5)
B" (2,5)
C" (0,4)

$T_{-1,6}$

(ii) A" (12,-6)
B" (18,-6)
C" (6,-12)

D_6

(iii) A" (-1,-2)
B" (-1,-3)
C" (-2,-1)

$Rot_{270°}$

LESSON: 55

AIM: How can we link geometric transformations with mathematical systems?

PERFORMANCE OBJECTIVES: The student will be able to...

1. State the properties of a group.
2. Determine the elements of an operation table where the elements are transformations and the operation is composition.
3. Determine whether or not a given set of transformations is a group under composition.

VOCABULARY: group, closure, associativity, identity element, inverse element, operation, set

MOTIVATION:

State the properties of a group (as in our study of mathematical systems from Course II).

$(G, *)$ is a group $\leftrightarrow$

1. G is closed under $*$
2. For any $a \in G$, $b \in G$, $C \in G$ $(a*b)*c = a*(b*c)$ associativity.
3. An identity element, $C \in G$, exists such that for any $A \in G$, $a*e = a$.
4. For each element $a \in G$, there exists an inverse for that element, $a^{-1} \in G$, such that $a*a^{-1} = e$.

DEVELOPMENT:

1. Consider the set of translations. What is the identity element in that set? The identity element in the set of translations, $T_{x,\, y}$, is $T_{0,\, 0}$.
   What operation can we use to illustrate that $T_{0,\, 0}$ is the identity translation? the operation we use is "composition" as in $T_{x,\, y} \circ T_{0,\, 0} = T_{x,\, y}$.

2. How can we determine if the set of translations under compositions is a group? We check each of the properties of a group. It is closed. I.e., $T_{x, y} \bullet T_{a, b} = T_{x+a, y+b}$. $T_{0, 0}$ is the identity element. $T_{-a, -b}$ is the inverse of $T_{a, b}$. What other transformations should we test for group properties? We can check reflections, rotations and dilations for group properties as well - each under composition.

Is reflections a group? No.

Why not? Reflections is not a group because

(i) It is not closed. We know that the composition of reflections can be a rotation or a translation.

Is rotations a group under composition? Yes. Why?

It is closed. $Rot_x \bullet Rot_B = Rot_{B+x}$

It is associative.

The identify element is Rot_0.

The inverse element of Rot_θ is $Rot_{-\theta}$

Is dilations under compositions a group if the scale factor is positive?

Yes, because

1. It is closed. $D_x \bullet D_y = D_{xy}$
2. It is associative.
3. The identity element is D_1.
4. The inverse element of D_k is $D_{1/k}$

   $(D_k, 0)$ is a group as long as $k > 0$.

*If we omit the restriction on the scale factor, what happens?

We lose the group property of existence of inverse for $k = 0$.

APPLICATIONS:

1. Fill in the following table:

o	I	R_y	R_x	$\mathrm{Rot}_{0,\ 180^\circ}$
I				
R_y				
R_x				
$\mathrm{Rot}_{0,\ 180^\circ}$				

2. Is each of the following a group? Verify.
   - (a) System in 1 above.
   - (b) (G, 0) where G = {Rotations about the origin}
   - (c) (G, 0) where G = {Positive dilations through the origin}.

SOLUTIONS TO APPLICATION PROBLEMS:

1. Choose a convenient test point: e.g., (2, 1)

$I\ (2, 1) = (2, 1)$

$R_y\ (2, 1) = (-2, 1)$ $\quad \mathrm{Rot}_{0,\ 180^\circ}(2, 1) = (-2, -1)$

$R_x\ (2, 1) = (2, -1)$

Since I leaves any point fixed, the first row and column match their headings.

Without loss of generality, we do the left following the top in the chart:

$R_y \bullet R_y\ (2, 1) = R_y\ (-2, 1) = (2, 1) \quad \therefore R_y \bullet R_y = I$

$R_y \bullet R_x\ (2, 1) = R_y\ (2, -1) = (-2, -1) \quad \therefore R_y \bullet R_x = \text{Rot}_{0,180^\circ}$

$R_y \bullet \text{Rot}_{0,\ 180^\circ}\ (2, 1) = R_y\ (-2, -1) = (2, -1) \quad \therefore R_y \bullet \text{Rot}_{0,180^\circ} = R_x$

$R_x \bullet R_y\ (2, 1) = R_x\ (-2, 1) = (-2, -1) \quad \therefore R_x \bullet R_y = \text{Rot}_{0,180^\circ}$

$R_x \bullet R_x = I$

$R_x \bullet \text{Rot}_{0,\ 180^\circ}\ (2, 1) = R_x\ (-2, -1) = (-2, 1) \quad \therefore R_x \bullet \text{Rot}_{0,\ 180^\circ} = R_y$

$\text{Rot}_{0,\ 180^\circ} \bullet R_y\ (2, 1) = \text{Rot}_{0,\ 180^\circ}\ (-2, 1) = (2, -1) \quad \therefore \text{Rot}_{0,180^\circ} \bullet R_y = R_x$

$\text{Rot}_{0,\ 180^\circ} \bullet R_x\ (2, 1) = \text{Rot}_{0,\ 180^\circ}\ (2, -1) = (-2, 1) \quad \therefore \text{Rot}_{0,\ 180^\circ} \bullet R_x = R_y$

$\circ$	I	R_y	R_x	$\text{Rot}_{0,\ 180^\circ}$
I	I	R_y	R_x	$\text{Rot}_{0,\ 180^\circ}$
R_y	R_y	I	$\text{Rot}_{0,\ 180^\circ}$	R_x
R_x	R_x	$\text{Rot}_{0,\ 180^\circ}$	I	R_y
$\text{Rot}_{0,\ 180^\circ}$	$\text{Rot}_{0,\ 180^\circ}$	R_x	R_y	I

2a. Clearly from the table, this group has closure, associativity, identity (I), and inverses. Each element is its own inverse.

2b and c are subgroups of groups studied in the lesson.

SUMMARY:

Which transformation sets form groups under composition?

Half-turns and Reflections fail because of closure. Dilations with positive scale factors, translations and rotations all succeed!

LESSON: 56

AIM: What is a glide reflection?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define a glide reflection as a composite of a translation and a reflection in a line parallel to the direction of the translation.
2. Demonstrate for a given diagram that the composition forming the glide reflection is commutative.
3. Find the image of a given set of points under a given glide reflection.
4. From a given diagram of an image and a pre-image, determine if there exists a glide reflection which maps the pre-image onto the image.

VOCABULARY: glide reflection

MOTIVATION:

Is the composition of line reflections commutative?
Illustrate your answer using
1. parallel lines: x = 2 and y - axis
2. intersecting lines: the axes

DEVELOPMENT:

1. Choose a convenient point (3,1)

$$R_{x=2}\,(3,\,1) = (1,\,1) \qquad\qquad R_y\,(3,\,1) = (-3,\,1)$$

$$\therefore R_y \quad R_{x\,=\,2}\,(3,1) = (-1,\,1) \qquad R_{x\,=\,2} \quad R_y\,(3,\,1) = (7,\,1)$$

$$\therefore R_y \quad R_{x\,=\,2} \neq R_{x\,=\,2} \quad R_y$$

$$R_x\,(3,\,1) = (3,\,-1) \qquad\qquad R_y\,(3,\,1) = (-3,\,1)$$

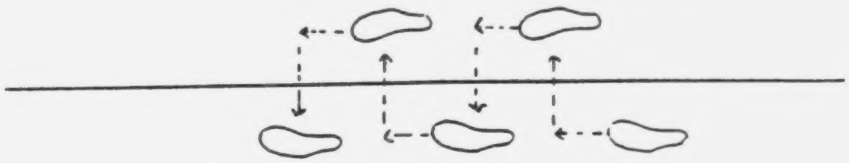
$$R_y \quad R_x\,(3,\,1) = (-3,\,-1) \qquad R_x \quad R_y\,(3,\,1) = (-3,\,-1)$$

$$\therefore R_y \quad R_x = R_x \quad R_y$$

Under certain conditions composition of transformations may be commutative.

2. Look back through your notes on composition of transformations. With the exception of yesterday's table, all of our examples had one common feature. What feature is that? All our previous problems (except for the table) have had the same type of transformation repeated in the problem.
   Today, we will look at a composition which deals with a line reflection and a translation in a direction parallel to the line. This type of composite transformation is called a **glide reflection**.

3. Draw a diagram of the footprints of a person walking forward in a straight line.



   This illustrates a glide reflection. Each step is reflected over a center line and then translated forward in a direction parallel to the line of reflection.
   Is this operation commutative?
   The diagram of the footprints seems to indicate so.
   We will verify this result in the subsequent problems.

APPLICATIONS:

1. The vertices of triangle ABC are A(-5, -2), B(-3, 1) and C(0, 4).
   (a) Graph triangle ABC and its image triangle A'B'C' under the translation T(4,0).
   (b) Graph triangle A"B"C", the image of triangle A'B'C' after a reflection in the y - axis.
   (c) Is the single transformation that maps triangle ABC into triangle A"B"C" a line reflection, point reflection, or glide reflection?
   (d) Re-do the problem. First apply a reflection in the y - axis and then apply $T_{4,0}$. What can we conclude about commutativity here?

2. Triangle BUG has coordinates B(1, 3), U(1, -3) and G(-1, 1).
   (a) Graph triangle BUG and its image, triangle B'U'G' after a reflection in the line whose equation is y= x.
   (b) Graph triangle B"U"G", the image of triangle B'U'G', after a translation $T_{3,3}$.
   (c) What term best describes the composite transformation?
   (d) Test for commutativity.
3. Given triangle ABC with coordinates A(4, 1), B(4, 5) and C(2, 5) and triangle A"B"C" with coordinates A"(-4, -5), B"(-2, -1) and C"(-4, -1). Describe a glide reflection which maps triangle ABC onto triangle A"B"C".

SOLUTIONS TO APPLICATION PROBLEMS:

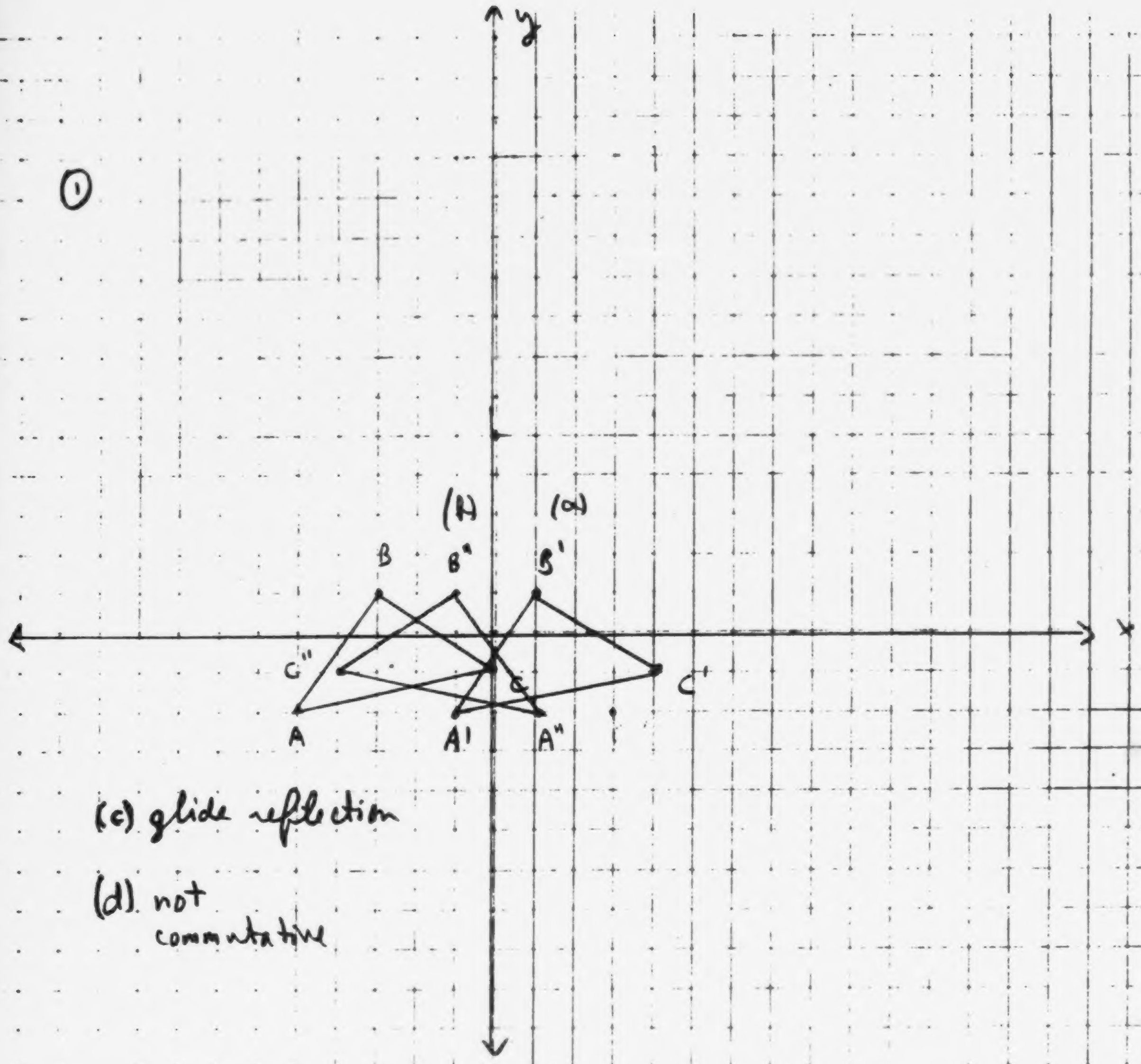
See attached graph paper.

SUMMARY:

How does the glide reflection fit into our study of transformations?

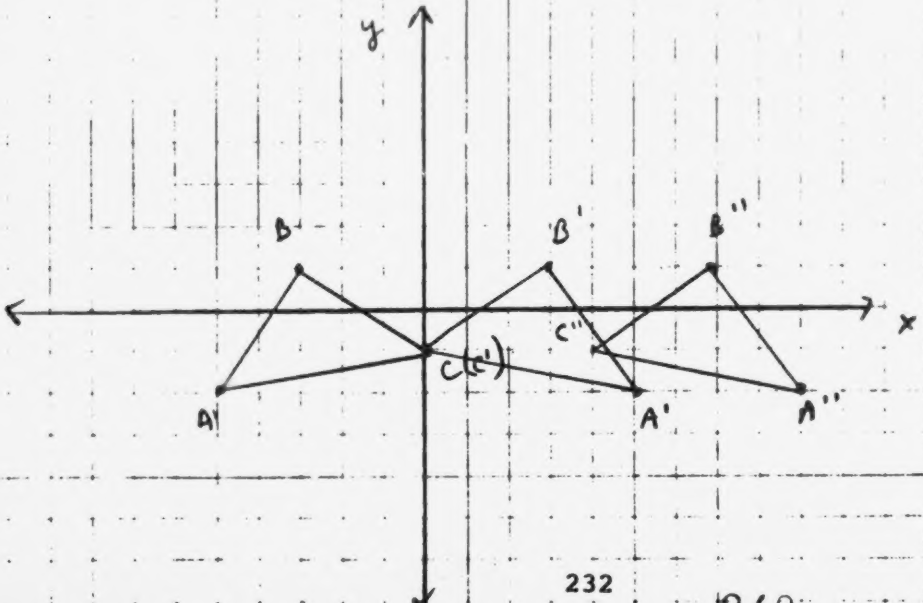
1. It is a composite of a translation and a line reflection where the line is parallel to the direction of the translation.
2. It is commutative.
3. It preserves distance and angle measure but not orientation.

①

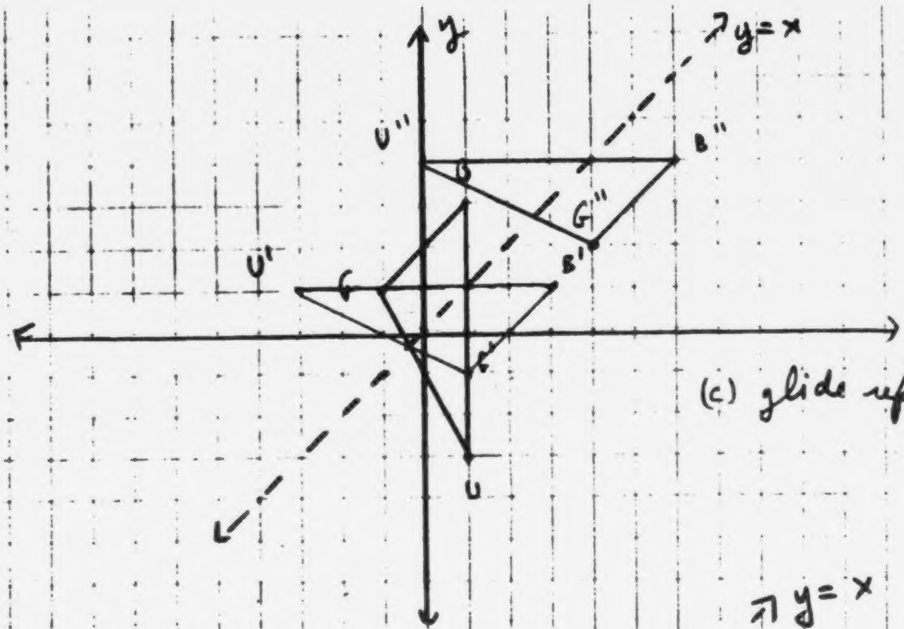

y
x
(b)
(a)
B
B''
B'
C''
C
C'
A
A'
A''

(c) glide reflection

(d) not commutative

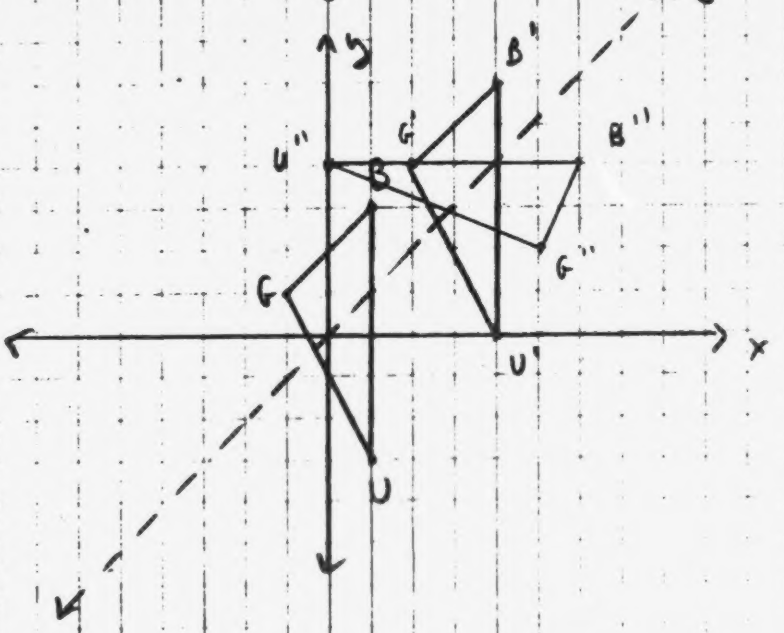

y
x
B
B'
B''
C''
C(C')
A
A'
A''

(21)

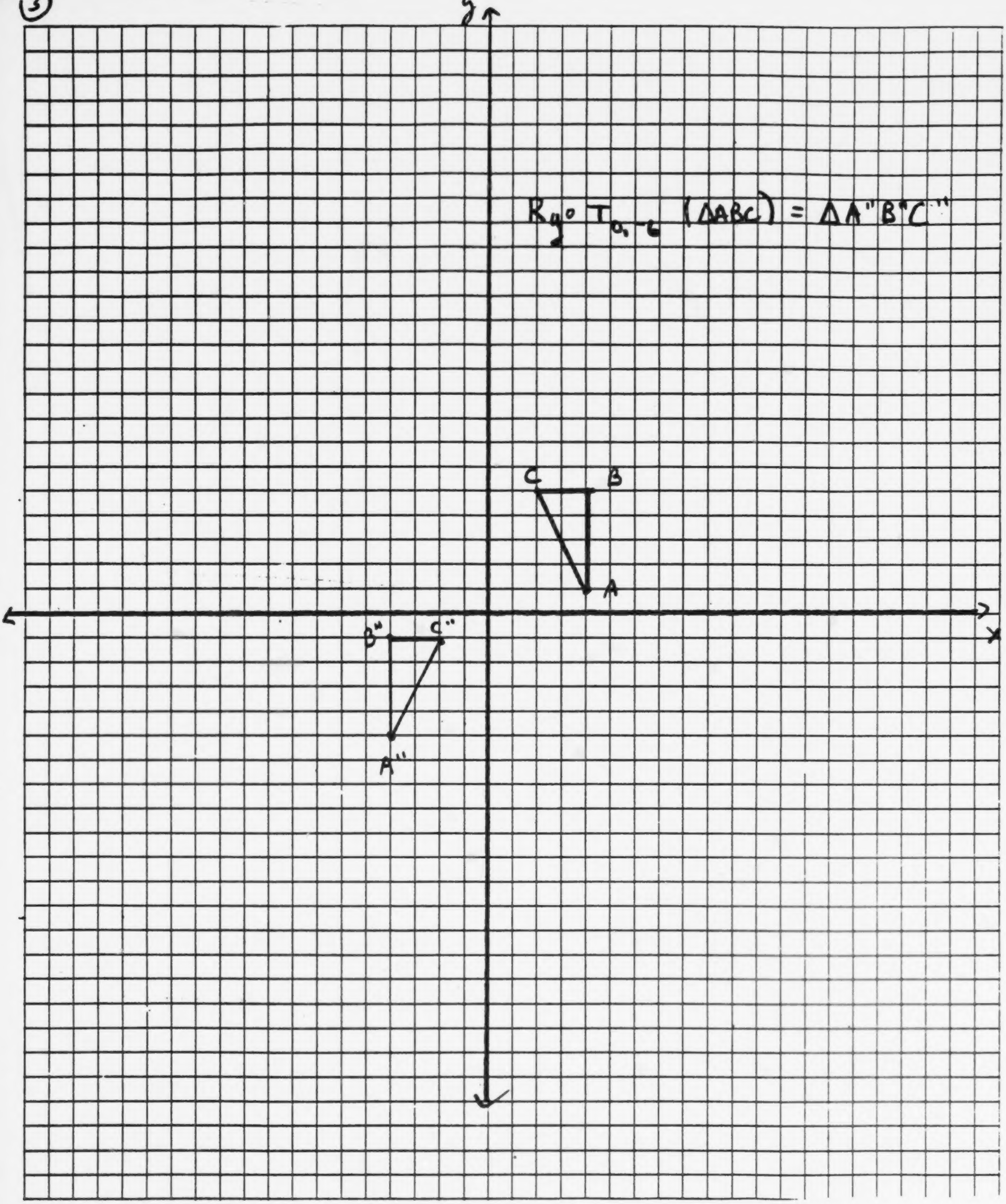

y
y = x
U''
B''
G''
B'
U'
G
x
G'
U

(c) glide reflection

(d)


y = x
y
B'
G'
B''
U''
B
G''
G
x
U'
U

③


y
x
$R_y \circ T_{0,-6}(\triangle ABC) = \triangle A''B''C''$
C
B
A
B''
C''
A''

LESSON: 57

AIM: What is an isometry?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Decide whether or not a given geometric transformation is an isometry.
2. Decide whether or not a given geometric transformation is a direct isometry.
3. Decide whether or not a given geometric transformation is an opposite isometry.

VOCABULARY isometry, direct isometry, opposite isometry

MOTIVATION:

An isometry is a distance-preserving transformation.

Which transformations which we have studied are isometries?
line reflections
rotations
half-turn (point reflections)
translations
glide reflections

DEVELOPMENT:

1. Which of the above isometries preserve orientation as well as distance?
   All of the above isometries except line reflections preserve orientation. Line reflections changes the orientation from clockwise to counterclockwise or vice-versa.
2. Explain to the class: Isometries which preserve orientation are direct isometries.
   Isometries which change orientation are opposite isometries.
   Is a glide reflection an example of a direct or opposite isometry?

A glide reflection is an opposite isometry because it involves one line reflection.

3. If S is a direct isometry and T is an opposite isometry, determine whether each composition is a direct or opposite isometry.

   1. $S \circ T$
   2. $T \circ S$
   3. $S \circ S$
   4. $T \circ T$
   5. $T \circ (T \circ T)$
   6. glide reflection

   Answers: direct: 3, 4
   opposite: 1, 2, 5, 6

   Each may be illustrated by applying the composition to triangle ABC.

   Result: The composite of two opposite isometries is a direct isometry.

   True/False? Every isometry may be expressed as the composition of at most 3 line reflections.

4. DISCUSSION: How can we break isometries down into cases?

   Case I: Line Reflections are trivially an illustration of a single line reflection.

   Case II: Rotations have been shown in an earlier lesson to be equivalent to the composition of two line reflections where the lines intersect in a point. Recall that the angle of rotation's measure is twice the measure of one of the angle's between the intersecting lines.

   Case III: Translations have been shown in an earlier lesson to be equivalent to the composition of two line reflections where the lines are parallel.

   Case IV: Since a glide reflection is a composition of a line reflection and a translation, and a translation is expressible as a composition of two line reflections, a glide reflection is equivalent to the composition of three line reflections.

What conclusion can be drawn? Elicit every isometry is expressible as a composition of at most three line reflections.

APPLICATIONS:

1. Classify each isometry as a line reflection, a glide reflection, a translation, or a rotation. Is the result a direct or opposite isometry?

   (a) $R_x\ (R_y)$

   (b) $R_x(R_{y=4})$

   (c) $R_y(T_{o,4})$

   (d) $T_{o,\ 4}(R_y)$

   (e) $\text{Rot}_{0,\ 180^{o}}(R_x)$

   (f) $R_x(\text{Rot}_{0,\ 270^{o}})$

   (g) $R_y = -x(R_{x=3})$

   (h) $R_x(R_y = x)$

2. Triangle ABC has vertices A(-1, 2), B(-3, 2), and C(-3, 0). Find the coordinates of:

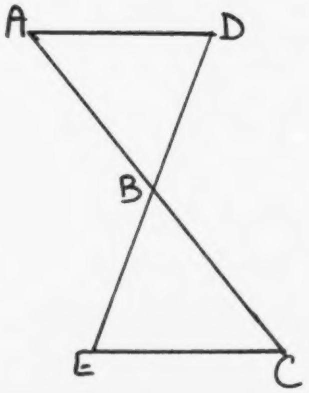
   (a) triangle A'B'C' where R_y(triangle ABC) = triangle, A'B'C'.

   (b) triangle A"B"C" where $R_{x=4}$(triangle A'B'C') = triangle A"B"C"

   (c) triangle A'''B'''C''' where $R_{x=-1}$(triangle A"B"C") = triangle A'''B'''C'''.

   (d) Show $R_y\ \ R_{x=4}\ \ R_{x=-1} = R_k$ by finding an equation of line k for which R_k (triangle ABC) = triangle A'''B'''C'''.

3. Complete this demonstration:



Given:

$\overline{AB} \cong \overline{BC}$

$\overline{DB} \cong \overline{BE}$

Prove:

Δ BAD $\cong$ Δ BCE

What single transformation maps A ⟶ C
D ⟶ E
B ⟶ B?

1. $\overline{AB} \cong \overline{BC}$, $\overline{DB} \cong \overline{BE}$ 1. Given

2.	$Rot_{B,\ 180^{o}}(\overline{AB}) =$ $Rot_{B,\ 180^{o}}(\overline{DB}) =$ $Rot_{B,\ 180^{o}}(\overline{AD}) =$	2.
3.	$Rot_{B,\ 180^{o}}$ is an isometry	
4.	$\triangle$ BAD $\cong$ $\triangle$ BCE	

SOLUTIONS TO APPLICATION PROBLEMS:

1. (a) rotation; direct isometry
   (b) translation; direct isometry
   (c) glide reflection; opposite isometry
   (d) glide reflection; opposite isometry
   (e) line reflection; opposite isometry
   (f) line reflection; opposite isometry
   (g) rotation; direct isometry
   (h) rotation, direct isometry

2. (a) A'(1, 2)
   B'(3, 2)
   C'(3, 0)

   (b) (7, 2)
   (5, 2)
   (5, 0)

   (c) (-9, 2)
   (-7, 2)
   (-7, 0)

   (d) k = -5
   because: $\frac{-9 + -1}{2} = -5$
   or $\frac{-7 + -3}{2} = -5$

3. $Rot_{B,\ 180^{0}}$

2.	$\overline{CB}$ $\overline{EB}$ $\overline{CE}$	2. Rotation maps segments to segments.
		3. A rotation is an isometry
		4. Since an isometry maps triangle BAD onto triangle BCE, the triangles are conguent.

SUMMARY:

How does the concept of isometry unify the concept of transformation?

Isometry brings together the line reflection, rotation, translation and glide reflection as distance-preserving transformations.

LESSON: 58

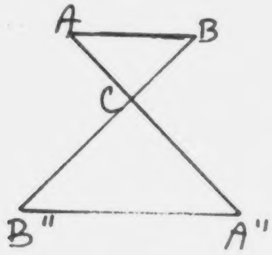
AIM: What is a similarity transformation?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Demonstrate that the composition of any isometry and a dilation produces a commutative similarity transformation.
2. State the ratio of similitude of similar polygons, given a similarity transformation or vice versa.
3. Find the lengths of sides of a polygon given the lengths of corresponding sides of another and a similarity transformation which maps one onto the other.

The word "commutative" is used here in the usual sense; however, the center of the dilations may be different as a result of reversing the order of the transformations. The scale factor will remain equal. See page 526 of Houghton Mifflin, Unified Mathematics - Book 3.

MOTIVATION:



AC = 4, BC = 3, CA" = 8. $\overline{AB}$ is parallel to $\overline{B"A"}$.

Suggest a composite transformation which maps triangle ABC into triangle A"B"C".

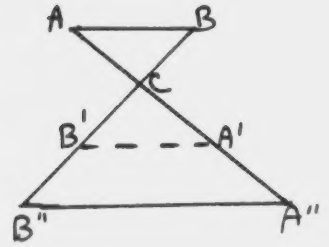
DEVELOPMENT:

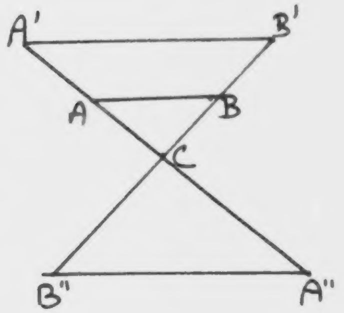
1. Is this an isometry? No! Why not? Because CA" ≠ AC. Therefore which type of transformation which we have studied must be involved? Dilation. Why? Because it is our only transformation which is not an isometry.

2. What appears to be the scale factor? 2 Why? Because $CA'' = 2 \cdot CA$.

3. How can we write a composition incorporating these ideas? Either

   1. $D_{C,2} \circ Rot_{C,\,180^\circ}$ or 2. $Rot_{C,\,180^\circ} \circ D_{C,2}$.





What is the relationship between the triangles? They are similar. I.e., triangle ABC $\sim$ triangle A"B"C" Why? Because $\angle A \cong \angle A''$, $\angle B \cong \angle B''$ (If 2 parallel lines are intersected by a transversal, the alt. int. $\angle$'s are =) and $\angle ACB \cong \angle B''CA''$ (vertical $\angle$'s are $\cong$.) Each composition is a similarity transformation. A similarity transformation is a transformation that is a composite of an isometry and a dilation.

4. What properties are preserved by a similarity transformation? Angle measure, betweenness, collinearity, parallelism. Distance is not preserved.

   Is the similarity transformation commutative? Yes. Why? It is commutative because the order of the transformation could be reversed but the resulting images are the same.

   How do we find B"C? Since the ratio of similitude is 2, $B''C = 2 \cdot BC = 2 \cdot 3 = 6$.

   OR $\frac{AC}{BC} = \frac{A''C}{B''C}$; $\frac{4}{3} = \frac{8}{x}$; $4x = 24$
   $x = 6$

APPLICATIONS:

Triangle R'S'T' is the image of triangle RST under the given similarity transformation.

a. if R'S' = 48, find RS   b. if ST = 3, find S'T'

1. $R_x \circ D_3$ 2. D_6 $T_{1,-2}$ 3. $Rot_{R,90^o} \circ D_{1/2}$

Triangle LMN has vertices L(3, 6), M(6, 3), N(9, 6). Find the coordinates of its image triangle under the given similarity transformation.

4. $R_y \circ D_{1/3}$ center is origin   5. $D_2 \circ R_x$ center is origin   6. $D_{1/3} \circ R_{y=x}$ $T_{-3,-3}$ center is origin

7. $R_x \circ D_5$ maps triangle ABC onto triangle A"B"C".
AB = 3, BC = 4 and AC = 5. Find A"B", B"C", A"C".

Express the similarity transformation that maps triangle RST onto triangle R"S"T" as a composite of an isometry following a dilation with center R.

8.	R(0, 0)	R"(0, 0)
	S(2, -1)	S"(-6, -3)
	T(1, -2)	T"(-3, -6)
9.	R(0, 0)	R"(2, 1)
	S(-1, 2)	S"(-1, 5)
	T(3, 1)	T"(7, 3)

SOLUTIONS TO APPLICATION PROBLEMS:

1. (a) $R'S' = 3 \cdot RS$ (b) $S'T' = 3 \cdot ST = 3 \cdot 3 = 9$
$48 = 3 \cdot RS$
$16 = RS$

2. (a) $R'S' = 6 \cdot RS$ (b) $S'T' = 6 \cdot ST = 6 \cdot 3 = 18$
$48 = 6 \cdot RS$
$RS = 8$

3. (a) $R'S' = \frac{1}{2} \cdot RS$ (b) $S'T' = \frac{1}{2} ST = \frac{1}{2} \cdot 3 = \frac{3}{2}$
$48 = \frac{1}{2} \cdot RS$
$96 = RS$

4.

L'(1, 2)	L"(-1, 2)
M'(2, 1)	M"(-2, 1)
N'(3, 2)	N"(-3, 2)

5.

L'(3, -6)	L"(6, -12)
M'(6, -3)	M"(12, -6)
N'(9, -6)	N"(18, -12)

6.

L'(0, 3)	L"(3, 0)	L'''(1, 0)
M'(3, 0)	M"(0, 3)	M'''(0, 1)
N'(6, 3)	N"(3, 6)	N'''(1, 2)

7. Since scale factor is 5

$$\frac{A''B''}{AB} = \frac{B''C''}{BC} = \frac{C''A''}{CA} = 5 \qquad \therefore \quad A''B'' = 15$$

$$B''C'' = 20$$

$$C''A'' = 25$$

$$\frac{A''B''}{3} = \frac{B''C''}{4} = \frac{C''A''}{5} = 5$$

8. From inspection of the coordinates, it seems $x'' = -3x$ and $y'' = 3y$.

We know that for D_3: $(x, y) \rightarrow (3x, 3y)$

and R_y: $(x, y) \rightarrow (-x, y)$

$\therefore$ $R_y \circ D_3$ (triangle RST) = triangle R"S"T

9. Again, inspection of the coordinates reveals.

$(x, y) \rightarrow (2x + 1, 2y + 1)$

$\therefore$ we use D_2: $(x, y) \rightarrow (2x, 2y)$

and $T_{1,1}$: $(x, y) \rightarrow (x + 1, y - 1)$

$\therefore$ $T_{1,1} \circ D_2$ (triangle RST) = triangle R"S"T".

SUMMARY:

How is similarity transformation different from an isometry? An isometry preserves distance but a similarity transformation involves the composition of an isometry and a dilation. Therefore, a similarity transformation does not preserve distance. A similarity transformation preserves angle measure, betweenness, collinearity, and parallelism!

LESSON: 59

AIM: What is a Sequence?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define "sequence".
2. Distinguish between finite and infinite sequences.
3. Determine a rule for finding the kth term of a sequence.

VOCABULARY: function, domain, range, sequence, term, general term, recursive

MOTIVATION:

Given the functions f and g with domain $\{1,2,3,4,5\}$. The function f is defined as $f(x) = 3x + 1$. The function g is defined as $g(1) = 4$ and $g(x+1)=g(x)+3$. Determine if these functions are equal.

DEVELOPMENT:

1. Ask: How do you find the range of each function? How can we determine if two functions are equal? Elicit from the class that for each function the range is the same and that each element in the domain is paired with the same element in the range, therefore the functions are equal. Tell the class that these functions are examples of special functions called sequences. Define sequence as follows:
   A SEQUENCE is a function having as its domain a set of positive integers.
2. Tell the class that instead of using the f(x) notation it is customary to use a_n, where n is used to represent a positive integer. Thus $f(x) = 3x + 1$ could be written as $a_n = 3n + 1$.
   Tell the class that the elements in the range of a sequence are called the terms of the sequence and are represented by $a_1, a_2, a_3, \ldots$The general term, or nth term,

of the sequence is a_n.

3. Point out that a sequence may be finite if the domain is the set $\{1,2,3,4,\ldots n\}$ or infinite if the domain is the set of positive integers.

4. Ask: How does the definition of f and g in the motivation differ?

   Elicit that f is defined explicitly and g is defined recursively.

   An explicit definition allows you to find a_n directly in terms of n.

   A recursive definition requires that you
   1) Identify the first term in the sequence, and
   2) State the relationship between each term and its successor.

APPLICATIONS:

I. Write the first five terms of each of the following sequences.

   1. $a_n = 5n + 3$
   2. $a_n = 2^n$
   3. $a_n = \frac{n(n-1)}{2}$
   4. $a_n = (-1)^n(n+3)$

II. Use the following recursive definitions to find the first five terms of the sequence. MAKE SURE STUDENTS UNDERSTAND EQUIVALENT WAY OF WRITING RELATIONSHIP BETWEEN EACH TERM AND ITS SUCCESSOR.

   5. $a_1 = 3,\ a_{n+1} = a_n - 2$
   6. $a_1 = 4,\ a_{n+1} = a_n + 5$
   7. $a_1 = -2,\ a_n = a_{n-1} + 2$
   8. $a_1 = 3,\ a_n = 3(a_{n-1})$
   9. $a_1 = 1,\ a_2 = 1,\ a_n = a_{n-1} + a_{n-2}$

III. For each sequence, find a rule for the nth term.

   10. 2,3,4,5,...
   11. 4,8,12,16,...
   12. 1/2,1/4,1/8,...
   13. 1,-1,1,-1,...

* IV. Present the Fibonacci Sequence by having the class consider the following:

A single pair of newly born rabbits is put into an enclosure at the beginning of January. Each pair of rabbits breeds a new pair at the beginning of the second month following birth and an additional pair at the beginning of each month thereafter. What is the number of pairs of rabbits at the beginning of September?

SOLUTIONS TO APPLICATION PROBLEMS:

I. Write out the first five terms of each of the following sequences.

1. $a_n = 5n+3$

$a_1 = 5(1)+3$
$= 8$
$a_2 = 5(2)+3$
$= 13$
$a_3 = 5(3)+3$
$= 18$
$a_4 = 5(4)+3$
$= 23$
$a_5 = 5(5)+3$
$= 28$

2. $a_n = 2^n$

$a_1 = 2^1 = 2$
$a_2 = 2^2 = 4$
$a_3 = 2^3 = 8$
$a_4 = 2^4 = 16$
$a_5 = 2^5 = 32$

3. $a_n = \frac{n(n-1)}{2}$

$a_1 = \frac{1(1-1)}{2} = 0$
$a_2 = \frac{2(2-1)}{2} = 1$
$a_3 = \frac{3(3-1)}{2} = 3$
$a_4 = \frac{4(4-1)}{2} = 6$
$a_5 = \frac{5(5-1)}{2} = 10$

4. $a_n = (-1)^n(n+3)$

$a_1 = (-1)^1(1+3) = -4$
$a_2 = (-1)^2(2+3) = 5$
$a_3 = (-1)^3(3+3) = -6$
$a_4 = (-1)^4(4+3) = 7$
$a_5 = (-1)^5(5+3) = -8$

II. Use the following recursive definitions to find the first five terms of the sequence.

5. $a_1 = 3$
$a_{n+1} = a_n - 2$
$a_1 = 3$
$a_2 = a_1 - 2$
$= 3-2$
$= 1$
$a_3 = a_2 - 2$
$= 1-2$
$= -1$
$a_4 = a_3 - 2$
$= -1-2$
$= -3$
$a_5 = a_4 - 2$
$= -3-2$
$= -5$

6. $a_1 = 4$
$a_{n+1} = a_n + 5$
$a_1 = 4$
$a_2 = a_1 + 5$
$= 4+5$
$= 9$
$a_3 = a_2 + 5$
$= 9+5$
$= 14$
$a_4 = a_3 + 5$
$= 14+15$
$= 19$
$a_5 = a_4 + 5$
$= 19+5$
$= 24$

7. $a_1 = -2$
$a_n = a_{n-1} + 2$
$a_1 = -2$
$a_2 = a_1 + 2$
$= -2+2$
$= 0$
$a_3 = a_2 + 2$
$= 0+2$
$= 2$
$a_4 = a_3 + 2$
$= 2+2$
$= 4$
$a_5 = a_4 + 2$
$= 4+2$
$= 6$

8. $a_1 = 3$
$a_n = 3(a_{n-1})$

$a_1 = 3$
$a_2 = 3a_1$
$= 3(3)$
$= 9$

$a_3 = 3(a_2)$
$= 3(9)$
$= 27$
$a_4 = 3(a_3)$
$= 3(27)$
$= 81$
$a_5 = 3a_4$
$= 3(81)$
$= 243$

9. $a_1 = 1$
$a_2 = 1$
$a_n = a_{n-1} + a_{n-2}$

$a_1 = 1$
$a_2 = 1$
$a_3 = a_1 + a_2$
$= 1+1$
$= 2$
$a_4 = a_3 + a_2$
$= 2+1$
$= 3$
$a_5 = a_4 + a_3$
$= 3+2$
$= 5$
Note: This is the Fibonacci Sequence.

III. For each sequence find a rule for the nth term.
Note: Questions should be done intuitively.
Show domain and ask class to guess a rule.

10. 2, 3, 4, 5,...

domain 1, 2, 3, 4

$a_n = n+1$

11. 4, 8, 12, 16,...

domain 1 2 3 4

$a_n = 4n$

12. 1/2, 1/4, 1/8,...

domain 1 2 3

$\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots$

$a_n = \frac{1}{2^n}$

13. 1, -1, 1, -1,...

domain 1 2 3 4

$a_n = (-1)^{n+1}$ or $a_n = (-1)^{n-1}$

* IV. Elicit that the sequence is 1,1,2,3,5,8,13,21,34. Challenge class to determine rule for finding the kth term (see 9 of Applications above).

SUMMARY:

What is meant by a sequence?

What is meant by the term of a sequence?

What conditions must be given in order to define a sequence recursively?

LESSON: 60

AIM: What is an Arithmetic Sequence?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Define an arithmetic sequence in terms of the constant difference.
2. Develop and apply the formula for the kth term.

VOCABULARY: arithmetic sequence, common difference

MOTIVATION:

John Smith is offered a position after graduating from college that starts at $22,000 with yearly raises of $2500. What is John's salary at the end of 6 years?

DEVELOPMENT:

1. Elicit from the class the following sequence.
   22,000, 24,500, 27,000 29,500, 32,000, 34,500
   Identify the above sequence as an ARITHMETIC SEQUENCE
   Definition
   An arithmetic sequence is a sequence in which each term after the first is obtained by adding the same fixed number to the previous term.
   The fixed number is called the common difference.
   Tell class that an arithmetic sequence is also called an arithmetic progression.
2. Do Application problems 1-12.
3. Challenge the class to develop the formula for the kth term by considering the following:

$a_1 = a_1$
$a_2 = a_1 + d$
$a_3 = a_2 + d = a_1 + d + d = a_1 + 2d$
$a_4 = a_3 + d = a_1 + 2d + d = a_1 + 3d$
$a_5 = a_4 + d = a_1 + 4d$
$a_6 = a_5 + d = a_1 + 5d$

How can we quickly arrive at the 15th term?

Elicit that any term can be found by the formula

$$a_n = a_1 + (n - 1)d$$

where a_n is the nth term, a_1 is the first term and d is the common difference.

4. Do Application problems 13-18.

APPLICATIONS:

I. Give the first five terms of each arithmetic sequence.

1. $a_1 = 3,\ d = 4$

$a_1 = 7,\ d = -2$

3. $a_1 = -2,\ d = 3$

4. $a_1 = 1/3,\ d = 2/3$

5. $a_1 = \sqrt{2},\ d = \sqrt{2}$

II. Determine whether or not the sequence is an arithmetic sequence. If it is an arithmetic sequence state the common difference and work a formula for a_n.

6. 15,18,21,24,...

7. 7,12,17,22,27,...

8. 18,6,2,2/3,...

9. 17,15,17,19,17,...

10. 20,17,14,11,...

11. $x,\ x+m,\ x+2m,\ x+3m,\ x+4m,\ \ldots$

12. $3x+y,\ 3x,\ 3x-y,\ 3x-2y,\ \ldots$

III. In each case, the first 3 terms of an arithmetic sequence are given. Find the indicated term.

13. -3,-1,1; 12th term

14. 7,4,1; 45th term

15. 1/4,3/4,5/4; 15th term

16. .02,.12,.22; 22 term

IV. Write a formula for a_n for each of the arithmetic sequences identified in II. (See answers in II above.)

17. What term of the arithmetic sequence 3,7,11, is 59?

18. Find three numbers in arithmetic sequence whose sum is 45 and whose product is 2640.

SOLUTIONS TO APPLICATION PROBLEMS:

I. Give the first five terms of each arithmetic sequence.

1. $a_1 = 3, d = 4$
$a_2 = 3+4 = 7$
$a_3 = 7+4 = 11$
$a_4 = 11+4 = 15$
$a_5 = 15+4 = 19$

2. $a_1 = 7, d = -2$
$a_2 = 7-2 = 5$
$a_3 = 5-2 = 3$
$a_4 = 3-2 = 1$
$a_5 = 1-2 = -1$

3. $a_1 = -2, d = 3$
$a_2 = -2+3 = 1$
$a_3 = 1+3 = 4$
$a_4 = 4+3 = 7$
$a_5 = 7+3 = 10$

4. $a_1 = 1/3, d = 2/3$
$a_2 = 1/3 + 2/3 = 1$
$a_3 = 1 + 2/3 = 5/3$
$a_4 = 5/3 + 2/3 = 7/3$
$a_5 = 7/3 + 2/3 = 3$

5. $a_1 = \sqrt{2}, d = \sqrt{2}$
$a_2 = \sqrt{2} + \sqrt{2} = 2\sqrt{2}$
$a_3 = 2\sqrt{2} + \sqrt{2} = 3\sqrt{2}$
$a_4 = 3\sqrt{2} + \sqrt{2} = 4\sqrt{2}$
$a_5 = 4\sqrt{2} + \sqrt{2} = 5\sqrt{2}$

II. Determine whether or not the sequence is an arithmetic sequence. If it is an arithmetic sequence state the common difference and work a formula for a_n.

6. 15,18,21,24,...
Yes: Difference of any two consecutive terms is a constant.
$18-15 = 24-21 = 3$
$a_n = a_1+(n-1)d$
$= 15+(n-1)(3)$
$= 15+3n-3$
$= 12+3n$

7. 7,12,17,22,27,...

Yes: Common difference is 5.

$$\begin{aligned} a_n &= a_1+(n-1)d \\ &= 7+(n-1)(5) \\ &= 7+5n-5 \\ &= 2+5n \end{aligned}$$

8. 18,6,2,2/3,...

No: Difference between consecutive terms is not constant.

$6-18 \neq 2-6$

9. 17,15,17,19,17,...

No: Difference between consecutive terms is not constant.

10. 20,17,14,11,...

Yes: Common difference is -3.

$$\begin{aligned} a_n &= a_1+(n-1)d \\ &= 20+(n-1)(-3) \\ &= 20+(n-1)(-3) \\ &= 20-3n+3 \\ &= 23-3n \end{aligned}$$

11. x, x+m, x+2m, x+3m, x+4m, ...

Yes: Common difference is m. $(x+m-x = x+2m-(x+m))$

$$\begin{aligned} a_n &= a_1+(n-1)d \\ &= x+(n-1)m \\ &= x+nm-m \end{aligned}$$

12. 3x+y, 3x, 3x-y, 3x-2y, ...

Yes: Common difference is -y.

$(3x - (3x+y) = 3x-y-3x)$

$$\begin{aligned} a_n &= a_1+(n-1)d \\ &= 3x+y+(n-1)(-y) \\ &= 3x+y-(n-1)y \end{aligned}$$

III. In each case, the first 3 terms of an arithmetic sequence are given. Find the indicated term.

13. $-3,-1,1$; 12th term

$a_1 = -3$
$n = 12$
$d = +2$
$a_n = a_1+(n-1)d$
$a_{12} = -3+(12-1)(2)$
$= -3+(11)(2)$
$= 19$

14. $7,4,1$; 45th term

$a_1 = 7$
$n = 45$
$d = -3$
$a_{45} = a_1+(n-1)d$
$= 7+(45-1)(-3)$
$= 7+(44)(-3)$
$= -125$

15. $1/4,3/4,5/4$; 15th term

$a_1 = 1/4$
$n = 15$
$d = 1/2$
$a_{15} = 1/4+(15-1)(1/2)$
$= 1/4+7$
$= 29/4$

16. $.02,.12,.22$; 22 term

$a_1 = .02$
$n = 22$
$d = .1$
$a_{22} = .02+(22-1)(.1)$
$= .02+(21).1$
$= .02+2.1$
$= 2.12$

IV. Write a formula for a_n for each of the arithmetic sequences identified in II. (See answers in II above.)

17. What term of the arithmetic sequence 3,7,11, is 59?

$a_n = 59$
$a_1 = 3$
$d = 4$
$n = ?$

$a_n = a_1+(n-1)d$
$59 = 3+(n-1)4$
$59 = 3+4n-4$
$59 = 4n-1$
$4n = 60$
$n = 15$

18. Find three numbers in arithmetic sequence whose sum is 45 and whose product is 2640.

Let $x-d$ = 1st term

x = 2nd term

$x+d$ = 3rd term

1) $x-d+x+x+d = 45$

2) $(x-d)(x)(x+d) = 2640$

From: 1) $3x = 45$

$x = 15$

Substituting in 2) $(15-d)(15)(15+d) = 2640$

$15(15^2-d^2) = 2640$

$15^2-d^2 = 176$

$d^2 = 49$

$d = \pm 7$

Ans: 8, 15, 22; $d = 7$

22, 15, 8; $d = -7$

SUMMARY:

1. What is an arithmetic sequence?
2. What do we mean by the common difference?
3. How can we find the common difference when we are given the terms of an arithmetic sequence?
4. What formula can be used to determine the nth term of any arithmetic sequence?

LESSON: 61

AIM: What is an arithmetic series.

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define series and distinguish series from sequence.
2. Develop and apply the formula for the sum of the arithmetic series.
3. Determine the arithmetic means between given terms.

VOCABULARY: arithmetic series, arithmetic means

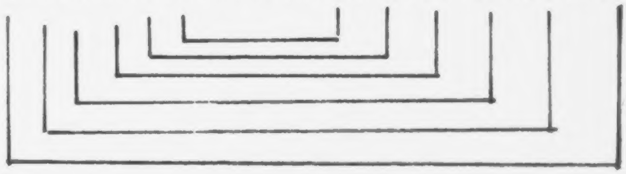
MOTIVATION:

Find the sum of the first 100 natural numbers.

DEVELOPMENT:

1. Give the students about 2 minutes to solve the problem. Relate this example to the story of young Gauss. If no student has found the answer tell the class that Gauss quickly arrived at the answer (5050) by pairing numbers in a certain way. Place the following on board and ask: How do you think he paired the numbers?

   $$S = 1+2+3+4+5+6+...94+95+96+97+98+99+100$$

   Suggest that he paired as shown above, there are 50 pairs so we multiply 50 by the sum of 100 and 1.
   Tell the class that the indicated sum of the terms of an arithmetic sequence is called an arithmetic series.

   STRESS. **SERIES IS THE INDICATED SUM OF THE TERMS OF A SEQUENCE NOT THE SUM ITSELF.**

2. Use the above as a model to derive the formula for the sum of the arithmetic series. Instead of a_n use ℓ for the last term.

$$S = a + a + d + a + 2d + \dots \ell$$
$$S = \ell + \ell - d + \ell - 2d + \dots a$$

Add both equations so that we have

$$2S = n(a+\ell)$$
$$S = \frac{n}{2}(a+\ell)$$

3. Do Applications 1 - 10.

4. Challenge the class to replace the blanks so that the resulting sequence is an arithmetic sequence.

   2,____,____,____,____,58

   Ask: What must we know in order to find the second term of the sequence?
   Elicit that we must know the common difference. We know that $a_1 = 2$, $a_5 = 58$, $n = 5$. Use the formula

   $$a_n = a_1 + (n-1)d$$
   $$a_5 = a_1 + (5-1)d$$
   $$58 = 2 + 4d$$
   $$d = 14$$

   Ask: How can we write the missing terms?
   Elicit that we must add the common difference to each term. Therefore the missing terms are 16, 30, 44.
   Tell the class that **the terms between any two terms of an arithmetic sequence are called ARITHMETIC MEANS between the two terms.**

5. Do Application 11. Be sure to point out in 11a that finding 1 arithmetic mean between two given numbers is equivalent to finding the average of the two numbers.

APPLICATIONS:

1. Find the sum of an arithmetic series of 22 terms whose first term is 5 and whose last term is 47.

2. Find the 16th term and the sum of the first 16 terms of the arithmetic series 2+8+14+20+...

3. Find the fourteenth term and the sum of the first fourteen terms of the arithmetic series 10+8+6+4+...

4. Find the sum of the first 400 terms of an arithmetic series beginning with 8 and whose common difference is 3.

5. Find the sum of the odd integers from 51 to 71.

6. The sum of the first 31 terms of an arithmetic series is 5580. If the thirty first term is 200, find the first term.

7. Mr. Smith wishes to build a wall with his son. The wall is constructed in the following way: The first layer contains 100 blocks. Each layer thereafter contains 3 less blocks until the last layer contains only 1 block. How many blocks will Mr. Smith need to build this wall?

8. The sum of the arithmetic series -9-5-1+3+7+... is 580. How many terms are in the series?

9. How many terms of the arithmetic series 24+22+20+... are needed to give a sum of 150.

10. Show that the sum of the first n odd positive integers is given by the formula $S = n^2$.

11. Find the missing terms in each of the arithmetic sequences.

    a. 40,____,106

    b. -9,____,____,____,____,-64

    c. ____,-7,____,____,14

SOLUTIONS TO APPLICATION PROBLEMS:

1. $s_{22} = ?$

$n = 22$

$a_1 = 5$

$a_n = \ell = 47$

$s_{22} = \frac{\overset{11}{\cancel{22}}}{\cancel{2}}(5+47)$

$= 11(52)$

$= 572$

2. $2+8+14+20+\ldots$

$a_1 = 2$

$n = 16$

$d = 6$

$a_{16} = a_1+(n-1)d$

$= 2+(16-1)6$

$= 2+(15)(6)$

$= 2+90$

$= 92$

$s_{16} = \frac{\overset{8}{\cancel{16}}}{\cancel{2}}(2+92)$

$= 8(94)$

$= 752$

3. $10+8+6+4+\ldots$

$n = 14$

$a_1 = 10$

$d = -2$

$a_{14} = 10+(14-1)(-2)$

$= 10+(13)(-2)$

$= 10-26$

$= -16$

$s_{14} = \frac{\overset{7}{\cancel{14}}}{\cancel{2}}(10+(-16))$

$= 7(-6)$

$= -42$

4. $n = 400$

$a_1 = 8$

$d = 3$

$a_{400} = 8+(400-1)(3)$

$= 8+(399)(3)$

$= 8+1197$

$= 1205$

$s_{400} = \frac{\overset{200}{\cancel{400}}}{\cancel{2}}(8+1205)$

$= 242600$

5. Find sum of odd integers from 51-71.

$n = 11$

$a_1 = 51$

$\ell = 71$

$s_{11} = \frac{11}{2}(51+71)$

$= \frac{11}{\cancel{2}}(\overset{61}{122})$

$= 671$

6. $s_{31} = 5580$

$a_{31} = 200$

$a_1 = ?$

$n = 31$

$s_{31} = \frac{n}{2}(a_1+\ell)$

$5580 = \frac{31}{2}(a_1+200)$

$5580(\frac{2}{31}) = a_1+200$

$a_1 = 160$

7. This is an arithmetic sequence of bricks. We seek the sum of an arithmetic series.

100+97+94+...1

We must know n. ‿an be found by using

$a_n = a_1+(n-1)d$

$1 = 100+(n-1)(3)$

$1 = 100+3n-3$

$3n = 96$

$n = 32$

$s_{32} = \frac{n}{2}(a_1+\ell)$

$= \frac{32}{2}(100+1)$

$= 16(101)$

$= 1616$

8. $s_n = \frac{n}{2}(a+\ell)$

$580 = \frac{n}{2}(-9+\ell)$

$\ell = a+(n-1)d$

$= -9+(n-1)(4)$

$= -9+4n-4$

$= 4n-13$

$580 = \frac{n}{2}(-9+4n-13)$

$= \frac{n}{2}(4n-22)$

$580 = n(2n-11)$

$2n^2-11n = 580$

$2n^2-11n-580 = 0$

$n = \frac{11 \pm \sqrt{121-4(2)(-580)}}{4}$

$n = \frac{11 \pm 69}{4}$

$n = \frac{80}{4}$ $n = \frac{-58}{4}$ reject

$n = 20$

9. $s = \frac{n}{2}(a+\ell)$ $\qquad\qquad \ell = a+(n-1)d$

$s = \frac{n}{2}(a_1+a_1+(n-1)d)$

$= \frac{n}{2}(2a_1+(n-1)d)$

$150 = \frac{n}{2}(2(24)+(n-1)(-2))$

$= \frac{n}{2}(48-2n+2)$

$= \frac{n}{2}(50-2n)$

$150 = n(25-n)$

$150 = 25n-n^2$

$n^2-25n+150 = 0$

$(n-15)(n-10) = 0$

$n = 15$ or $n = 10$

both are correct

10. $1+3+5+\ldots 2n-1$

$s_n = \frac{n}{2}(a+\ell)$

$= \frac{n}{2}(1+2n-1)$

$= \frac{n}{2}(2n)$

$= n^2$

11. Method I

a) $40, x, 106$

$106-x = x-40$

$2x = 146$

$x = 72$

Method II

We seek the arithmetic mean between 40 and 106.

Arithmetic mean $= \frac{40 + 106}{2}$

$= 72$

b) Must find d, can use formula $a_n = a_1+(n-1)d$

$-64 = -9+(6-1)d$

$-55 = 5d$

$d = -11$

$-9, \underbrace{-20, -31, -42, -53}_{\text{Arithmetic means}}, -64$

c) ____, -7, ____, ____, 14

$a_n = a_1 + (n-1)d$ $\qquad$ Note: Consider -7 first term.

$14 = -7+(4-1)d$

$14 = -7+3d$

$3d = 21$

$d = 7$

$-14, -7, 0, 7, 14$

SUMMARY:

1. What is an arithmetic series?
2. How does an arithmetic sequence and arithmetic series differ?
3. What is the formula for finding the sum of an arithmetic series?
4. Define arithmetic mean.

LESSON: 62

AIM: What is a geometric sequence?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define a geometric sequence in terms of the common ratio.
2. Develop and apply the formula for the kth term.

VOCABULARY: geometric sequence, common ratio

MOTIVATION:

For saving his daughter from a dragon, an ancient king asked the brave knight to name his own reward. The knight replied: "I would like some wheat for each of the next 64 days. On day 1 I would like 1 grain, on day 2, 2 grains, on day 3, 4 grains, doubling the previous days amount until the 64th day is reached. The king said "Is that all?" The knight smiled and said "Yes sire, that is all." WHY DID THE KNIGHT MAKE A SMART BARGAIN?

DEVELOPMENT:

1. Elicit from the class that the sequence for the amount of grain would be as follows:

   1,2,4,8,16,32,64,...

   Ask: How is this sequence being formed?

   Elicit that each term is formed by multiplying the previous term by 2.

   Ask the class to determine a rule for finding the sequence.

   Identify such a sequence as a geometric sequence.

   GEOMETRIC SEQUENCE (PROGRESSION) IS A SEQUENCE IN WHICH EACH TERM AFTER THE FIRST IS OBTAINED BY MULTIPLYING THE PRECEDING TERM BY A CONSTANT NON-ZERO REAL NUMBER, CALLED THE COMMON RATIO.

   Define the common ratio as $r = \frac{a_{n+1}}{a_n}$

2. Using motivation and the following, generalize to find the formula for the kth term.

   Assume that the sequence has a first term (a_1) and common ratio, r. Then

   $$a_1 = a_1$$
   $$a_2 = a_1 r$$
   $$a_3 = (a_1 r)r = a_1 r^2$$
   $$a_4 = (a_1 r^2)r = a_1 r^3$$

   Ask: If this pattern would continue, how could you quickly find the exponent of r for any term of this sequence?

   Elicit that the exponent of r is k - 1 and that the kth term is

   $$a_k = a_1 r^{k-1}$$

APPLICATIONS:

1. Express in exponential form the 30th term of a geometric sequence whose first term is -9 and whose common ratio is 3.
2. Find in exponential form the 20th term of the geometric sequence 4,16,64,...
3. Find the 7th term of the geometric sequence 9,-6,4,...
4. Find the common ratio for a geometric sequence whose first term is 9 and whose 6th term is 9216.
5. Find the tenth term of the geometric sequence $\sqrt{3}, 1, \frac{\sqrt{3}}{3}, \ldots$
6. The third term of the geometric sequence is 12 and the sixth term is 96. Find the first term.
7. A type of bacterium divides by binary fission every 30 minutes. How many bacteria would a single organism become at the end of 4 hours, assuming that all survived and that they continued dividing every 30 minutes.
8. The first term of a geometric sequence, with a common ratio 1/2, is 8. The last term is 1/8. How many terms are in the sequence?

SOLUTIONS TO APPLICATION PROBLEMS:

1. $a_n = a_1 r^{n-1}$
$a_{30} = (-9)(3)^{30-1}$
$= (-9)(3)^{29}$
$= (-1)(3^2)(3^{29})$
$= (-1)3^{31}$

2. $n = 20$
$a_1 = 4$
$r = 4$ since $\frac{16}{4} = \frac{64}{16} = 4$
$a_{20} = 4(4)^{20-1}$
$= 4(4^{19})$
$= 4^{20}$

3. $n = 7$
$a_1 = 9$
$r = \frac{-2}{3}$ since $\frac{-6}{9} = \frac{4}{-6} = \frac{-2}{3}$
$a_7 = 9(\frac{-2}{3})^7$
$= 9\frac{(-2)^7}{3^7}$
$= \frac{(-2)^7}{3^5}$
$= \frac{-128}{243}$

4. $a_1 = 9$
$a_6 = 9216$
$r = ?$
$a_6 = a_1 r^{n-1}$
$9216 = 9r^{6-1}$
$\frac{9216}{9} = r^5$
$r^5 = 1024$
$r = 4$

5. $a_1 = \sqrt{3}$
$n = 10$
$r = \frac{1}{\sqrt{3}}$
$a_{10} = \sqrt{3}(\frac{1}{3})^{n-1}$
$= \sqrt{3}(\frac{1}{3})^9$
$= 3^{1/2}\left(\frac{1}{3^{1/2}}\right)^9$
$= \frac{3^{1/2}}{3^{9/2}} = \frac{1}{3^4} = \frac{1}{81}$

6. $a_3 = 12$ Must find r.
$a_6 = 96$ Since any
$a_1 = ?$ term can be
$a_n = a_1 r^{n-1}$ considered
$96 = 12r^{4-1}$ proceed as
$8 = r^3$ follows.
$r = 2$ Note: $n = 4$

$\therefore a_2 = 6$
$a_1 = 3$

7. $a_1 = 1$ Point out why n must be 9.
$r = 2$ At time $t = 0$, $a_1 = 1$, $t = 30$ min, $a_2 = 2$,
$n = 9$
$a_n = a_1 r^{n-1}$
$= (1)(2)^8$
a^n
$a_9 = 256$

8. $a_1 = 8$

$r = \frac{1}{2}$

$a_n = 1/8$

$a_n = a_1 r^{n-1}$

$1/8 = 8(\frac{1}{2})^{n-1}$

$\frac{1}{64} = (\frac{1}{2})^{n-1}$

$(\frac{1}{2})^6 = (\frac{1}{2})^{n-1}$

$6 = n-1$

$n = 7$

Excellent place to review exponential equations.

SUMMARY:

1. What is a geometric sequence?
2. How do we determine the common ratio for a given geometric sequence?
3. Why can't any term of a geometric sequence be 0?
4. In a geometric sequence why can't all the terms by negative?
5. In a geometric sequence why can the terms alternate in sign?

LESSON: 63

AIM: What is a geometric series?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Develop and apply the formula for the sum of a geometric series.
2. Determine the geometric means between given terms.

VOCABULARY: geometric series, geometric means

MOTIVATION:

Overconfident Sam challenges quiet Sheldon to a golf match and proposes that they play for the following stakes. The first hole would be worth 5 cents, the second hole 10 cents, the third hole 20 cents, and so on, for the whole 18 holes. Overconfident Sam lost every hole. How much did he owe?

DEVELOPMENT:

1. Have class write out the sequence of wagers for each hole (5,10,20,40,...). Elicit that this problem requires that we determine the sum of all the terms rather than the last term. Identify this sum as a geometric series.

   GEOMETRIC SERIES IS THE INDICATED SUM OF THE TERMS OF A GEOMETRIC SEQUENCE

2. Challenge class to derive the formula for the sum of the geometric series.

   $S_n = a + a_1r + a_1r^2 + a_1r^3 + \ldots a_1r^{n-1}$

   Remind students that in deriving sum of arithmetic series we wrote the series in reverse order and then added. This procedure will not work in this example. Suggest that multiplying each term of this sequence by r will result in a sequence with many terms that are the same as the original sequence.

   $$\begin{aligned} S_n &= a_1 + a_1r + a_1r^2 + a_1r^3 + \ldots + a_1r^{n-1} \\ rS_n &= \phantom{a_1 + {}} a_1r + a_1r^2 + a_1r^3 + \ldots + a_1r^{n-1} + a_1r^n \end{aligned}$$

How can we combine these two equations so that some of the terms will be eliminated? (Subtract them.)

$S_n - rS_n = a_1 - a_1r^n$

In order to get S_n alone we must factor the left side.

$S_n(1-r) = a_1 - a_1r^n$

or

$$S_n = \frac{a_1 - a_1r^n}{1 - r};\ r \neq 1$$

Go back and find the answer to the motivational problem.

$$\begin{aligned} S &= \frac{5 - 5(2^{18})}{1 - 2} \\ &= 5(2^{18} - 1) \\ &= 5(262144) \\ &= 1310720 \text{ cents} \\ &= \$13,107.20 \end{aligned}$$

3. Do Applications 1-5.

4. Challenge the class to fill in the blank so that the resulting numbers form a geometric sequence.

   4, ___x___, 9

   Ask: How can we express the common ratio in terms of x?

   Elicit that $\frac{9}{x} = \frac{x}{4}$

   $x^2 = 36$

   $x = \pm 6$

5. Tell class that **the terms between any two given terms of a geometric sequence are called geometric means.**

   A single geometric mean between two given terms of a geometric sequence is called the geometric mean between the two terms (also called the mean proportional).

6. Derive general formula for geometric mean between 2 numbers a and b.

   geometric mean = $\sqrt{ab}$

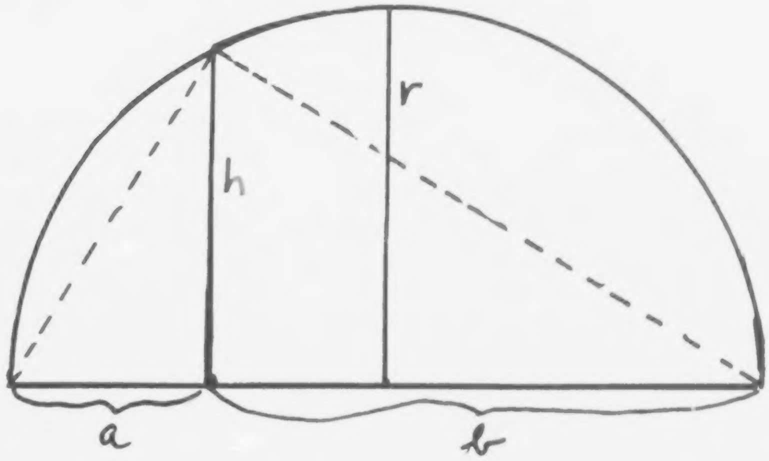
7. Do Applications 6, 7.

APPLICATIONS:

1. Express in exponential form the sum of the first 30 terms of the geometric series 3+6+12+24+...
2. Find the sum of the first ten terms of the geometric series -4+8-16+32+...
3. Find the sum of the first five terms of a geometric sequence whose first term is 1 and whose common ratio is 1/3.
4. Find the first term of a geometric series of 8 terms whose sum is 1020 and whose common ratio is 2.
5. Express in exponential form the sum of the first 15 terms of the geometric series.

   $$\log 7 + \log 7^2 + \log 7^4 + \log 7^8 + \ldots$$

6. Insert three geometric means between 1/9 and 9.
7. Let a and b be the lengths of a pair of adjacent segments on a line, and draw a semicircle with the two segments together as its diameter (see diagram below). Show that the radius of the circle is the arithmetic mean of a and b, and that the perpendicular distance h is their geometric mean. Show that the arithmetic mean $\geq$ geometric mean. When will the equality hold?



SOLUTIONS TO APPLICATION PROBLEMS:

1. $S_n = \frac{a_1 - a_1 r^n}{1-r}$

$n = 30$

$a_1 = 3$

$r = 2$

$$S_{30} = \frac{3-3(2)^{30}}{1-2}$$

$$= 3(2^{30}-1)$$

2. $S_n = \frac{a_1 - a_1 r^n}{1-r}$

$n = 10$

$a_1 = -4$

$r = -2$

$$s_{10} = \frac{-4-(-4)(-2)^{10}}{1-(-2)}$$

$$= \frac{-4(1+(-2)^{10})}{3}$$

3. $S_5 = \frac{1-1(\frac{1}{3})^5}{1-\frac{1}{3}}$

$$= \frac{1-\frac{1}{243}}{\frac{2}{3}}$$

$$= \frac{(243-1)/243}{2/3}$$

$$= \frac{242}{243} \cdot \frac{3}{2} = \frac{121}{81}$$

4. $S_n = \frac{a_1 - a_1 r^n}{1-r}$

$a_1 = ?$

$s_8 = 1020$

$r = 2$

$$1020 = \frac{a_1 - a_1 (2)^8}{1-2}$$

$$1020 = a_1(2^8-1)$$

$$a_1 = \frac{1020}{255}$$

$$a_1 = 4$$

5. $S_n = \frac{a_1 - a_1 r^n}{1-r}$

$a_1 = \log 7$

$n = 15$

$r = 2$

$$s_{15} = \frac{\log 7-(\log 7)2^{14}}{1-2}$$

$$= 2^{14} \log 7 - \log 7$$

$$= \log 7(2^{14}-1)$$

6. Insert 3 geometric means between 1/9 and 9.
Need to find r. Sequence will have 5 terms.

$$a_n = a_1 r^{n-1}$$

$$9 = \frac{1}{9} r^{5-1}$$

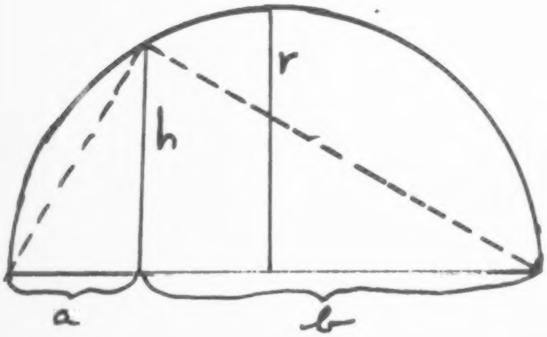
$$81 = r^4$$

$$r = \underline{+}3$$

If r = 3, Ans. $\frac{1}{9}, \underbrace{\frac{1}{3}, 1, 3}_{\text{Geometric means}}, 9$

$r = -3$ $\quad \frac{1}{9}, \underbrace{\frac{-1}{3}, 1, -3}_{\text{Geometric means}}, 9$

7.



1. The length of the diameter is a+b. Therefore the length of radius is $\frac{a+b}{2}$, which is the arithmetic mean of a and b.

2. H is an altitude drawn to hypotenuse of a right triangle. Therefore, h is the mean proportional between the seqments of the hypotenuse.

$$\frac{a}{h} = \frac{h}{b}$$

$h = \sqrt{ab}$ which is the geometric mean between a and b.
Clearly the radius must be greater than h.

Therefore $\frac{a+b}{2} > \sqrt{ab}$

If a and b were equal then h=r and we have that

$$\frac{a+b}{2} = \sqrt{ab}.$$

SUMMARY:

1. What is a geometric series?
2. What is the formula for finding the sum of a geometric series?
3. How would you define geometric means?
4. What is another name for the geometric mean between 2 numbers.

LESSON: 64

AIM: How do we find the sum of an infinite geometric series?

PERFORMANCE OBJECTIVES: The student will be able to:

1. Develop from the formula $S_n = \frac{a - ar^n}{1-r}$ the formula for the infinite geometric series.
   $S = \frac{a}{1 - r}$; $|r| < 1$
2. Apply this formula to problems, including repeating decimals.

VOCABULARY: infinite geometric series, repeating decimals, limit

MOTIVATION:

Mountain Climbing Sam wants to climb a mountain that is 2 miles high. He estimates that he can climb 1 mile the first day. After that the climb gets harder, so he estimates that on day 2 he can climb 1/3 of a mile, and on day 3, 1/9 of a mile, each day climbing 1/3 of the previous amount. How long will it take him to reach the top?

DEVELOPMENT:

1. Have the class write out the conditions of the problem as $1 + 1/3 + 1/9 + 1/27 + \ldots$

   Ask questions such as the following to elicit that we must find the sum of the series.

   1. What kind of series is this?
   2. What is the common ratio?
   3. What must we find in order to solve this problem?

   Have the class find the sum(S_n) for a given number of terms. Assign a different sum to each row.

   Find S_3, S_4, S_5, S_{10}, S_{20}

   Ask: What appears to be true about the sequence of sums? Elicit that they are increasing but never reach 3/2? What happens as we take more and more terms? (Allow n to become very large.)

Elicit that the value gets very small and it approaches 0. Define $\lim_{n \to \infty} r^n = 0$ if $|r| < 1,\ r \neq 0$.

2. Develop formula for the Infinite Geometric Series.

$$S = \frac{a_1 - a_1 r^n}{1-r}$$

$$S = \frac{a_1}{1-r} - \frac{a_1 r^n}{1-r}$$

What will happen to the value of the fraction $\frac{a_1 r^n}{1-r}$ as n gets very large?

Elicit that it will approach zero.

Therefore the sum of an infinite geometric series is given as:

$$S = \lim_{n \to \infty} S_n = \frac{a}{1 - r}$$

3. Do Application problems 1-5,9,10.

4. Challenge the class to write the repeating decimal .3333... as a rational number in the form p/q.

   Questions such as the following should be used to elicit the general method.

   1. How can we write .3333... as a geometric series?
   2. What is the common ratio?
   3. How can we find the sum of this infinite geometric series?

   Method

   1. Find the first term.
   2. Find the common ratio.
   3. Use the formula $S = \frac{a}{1-r}$ to find the sum.

5. Do Applications 6,7,8,11,12.

APPLICATIONS:

I. Find the sum, if possible, of each of the following infinite geometric series.

1. $-1/12 - 1/6 - 1/3 - 2/3 + \ldots$
2. $15 + 5 + 5/3 + 5/9 + \ldots$
3. $1 - 2/3 + 4/9 - 8/27 + \ldots$
4. $1 - 2 + 4 - 8 + \ldots$
5. $1/10 + 1/100 + 1/1000 + 1/10000 + \ldots$

II. Express the repeating decimal in the form p/q.

6. 0.12121212...
7. 0.90909090...
8. 0.5688888...

III.

9. You are 4 ft from a wall and walk towards the wall with each step covering half the remaining distance. Represent the length of each step with a series and find the sum.
10. A sequence of equilateral triangles is constructed as follows: The second triangle is formed by connecting the midpoints of the sides of the original equilateral triangle. The third is constructed by connecting the midpoints of the sides of the second triangle. If the process is continued indefinitely, what will be the total perimeter of the triangles if the side of the original triangle was 2 meters.
11. A rubber ball is dropped from a height of 27 feet. On each bounce the ball returns to 2/3 of its previous height. About how far will the ball travel before it comes to rest. (Hint: Have class draw a diagram and consider sum of 2 infinite geometric series.)
12. The sum of an infinite geometric series is 3 and its first term is $\sqrt{2}$. Find the common ratio.

SOLUTIONS TO APPLICATION PROBLEMS:

I.

1. $-1/12-1/6-1/3-2/3+\ldots$

$$r = \frac{\frac{-1}{6}}{\frac{-1}{12}} = 2$$

Since $|r|$ is not less than 1. This sequence has no sum.

2. $15+5+5/3+5/9+\ldots$

$r = 1/3$

Since $|r| < 1$ we use formula

$$S = \frac{a}{1-r}$$

$$= \frac{15}{1-1/3}$$

$$= \frac{15}{2/3} = \frac{45}{2}$$

3. $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27}$

$r = -2/3$

$$S = \frac{a}{1-r}$$

$$= \frac{1}{1-(-\frac{2}{3})}$$

$$= \frac{1}{1+2/3}$$

$$= \frac{3}{5}$$

4. $1-2+4-8+\ldots$

$r = -2$

Since $|r|$ is not less than 1 this sequence has no sum.

5. $\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} + \ldots$

$a = \frac{1}{10}$

$r = \frac{1}{10}$

$$S = \frac{\frac{1}{10}}{1-\frac{1}{10}} = \frac{\frac{1}{10}}{\frac{9}{10}} = \frac{1}{9}$$

II.

6. $0.12121212\ldots$

$$S = \frac{12}{100} + \frac{12}{10000} + \frac{12}{1000000}$$

$r = .01$

$a = .12$

$$S = \frac{.12}{1-.01}$$

$$= \frac{12}{99}$$

$$= \frac{4}{33}$$

7. 0.90909090

$r = .01$

$a = .90$

$$S = \frac{.90}{1-.01}$$

$$= \frac{.90}{.99}$$

$$= \frac{90}{99}$$

$$= \frac{10}{11}$$

8. 0.5688888...

$$.56 + \underbrace{\frac{8}{1000} + \frac{8}{10000} + \frac{8}{100000}}_{\text{Infinite Geometric Series}} + \ldots$$

$$S = \frac{\frac{8}{1000}}{1-.1} = \frac{\frac{8}{1000}}{\frac{9}{10}} = \frac{8}{900}$$

$$.56 + \frac{8}{900} = \frac{56}{100} + \frac{8}{900}$$

$$= \frac{512}{900} = \frac{128}{225}$$

9. $2 + 1 + \frac{1}{2} + \frac{1}{4} + \ldots$

$r = \frac{1}{2}$, $a = 2$

$$S = \frac{a}{1-r}$$

$$= \frac{2}{1-\frac{1}{2}}$$

$$= 4$$

10. Perimeter of original triangle is 6.

The infinite geometric sequence is

$6+3+\frac{3}{2}+\frac{3}{4}+\ldots$

$$S = \frac{a}{1-r}$$

$$= \frac{6}{1-\frac{1}{2}}$$

$$= \frac{6}{1/2}$$

$$= 12$$

11. $27\downarrow 18\uparrow 18\downarrow 12\uparrow 12\downarrow$

$27+18+12+\ldots$

$18+12+8+\ldots$

$$S = \frac{27}{1-\frac{2}{3}} = 81$$

$$S = \frac{18}{1-2/3} = 54$$

Total distance: 135

12. $S = \sqrt{3}$

$a = \sqrt{2}$

$r = ?$

$$S = \frac{a}{1-r}$$

$$\sqrt{3} = \frac{\sqrt{2}}{1-r}$$

$$(1-r)\sqrt{3} = \sqrt{2}$$

$$1-r = \frac{\sqrt{2}}{\sqrt{3}}$$

$$r = 1 - \frac{\sqrt{2}}{\sqrt{3}}$$

$$= \frac{\sqrt{3} - \sqrt{2}}{\sqrt{3}}$$

SUMMARY:

1. Define a infinite geometric series.
2. Give an example of an infinite geometric series that does not have a sum.
3. Explain the conditions necessary to insure that an infinite geometric series can have a sum.
4. Explain how the formula $S = a/(1-r)$ is obtained from the formula $S = a_1(1-r^n)/(1-r)$.

LESSON: 65

AIM: How do we prove that a formula always works?

PERFORMANCE OBJECTIVES: The student will be able to...

1. Define the principle of mathematical induction.
2. Apply the principle to simple proofs involving formulas.

VOCABULARY: induction, mathematical induction

MOTIVATION:

Sam challenged John to find the sum of the first 100 odd integers. John replied that is easy. The sum is equal to n^2. Prove that John's formula is correct?

DEVELOPMENT:

1. Elicit that we must show that $1+3+5+7+\ldots = n^2$.

   Elicit that the truth of this formula can be checked quickly for the first few values of n. We could do this indefinitely, but we would still not have proved the formula true for every positive integer.

2. Tell class that to prove a statement true for all positive integers we use the **Principle of Mathematical Induction (PMI).**

   Let S_n be a statement concerning positive integer n. If

   1. S_1 is true and
   2. when S_n is true for some positive integer k, then S_n is also true for the integer k + 1.

   then S_n is true for every positive integer value of n.

   In other words we must

   1. show that formula is true for n=1
   2. assume that formula is true for n=k and show that formula is therefore true for n = k + 1.

Compare the method to that of dominoes lined up so that if you push the first dominoe they all fall down. That is to say, two things are necessary for the dominoes to fall

1. the first dominoe must fall
2. they must be set up in such a way that if one falls, they all do.

These necessary conditions should be compared to the two requirements of PMI.

3. Show how PMI can be used to establish formula from motivation.

Let S_n be set of natural numbers for which

$1+3+5+7+...2n-1 = n^2$

1) Show that $1 \in S_n$

$1 = 1$

2) Assume true for k

$1 + 3 + 5 + 7 + 2k-1 = k^2$

We must show that formula is true for k+1 term. That is to say

$1 + 3 + 5 + 7 +...2k-1 + 2k+1 = (k+1)^2$

Start with $1 + 3 + 5 +...2k-1 = k$ and add $2K + 1$ to both sides

$1 + 3 + 5 +...2k-1 + 2k+1 = k^2 + 2k + 1$

$1 + 3 + 5 +...+ 2k-1 + 2k+1 = (k + 1)^2$

Since we have shown that the formula is true for k+1 we can conclude that the formula is true for all n.

4. Use the Principle of Mathematical Induction to show that

$1 + 2 + 3 + 4 +...n = \frac{n(n+1)}{2}$

Let S_n be set of natural numbers for which

$1+2+3+...n = \frac{n(n+1)}{2}$

Step 1. Show that $1 \in S_n$

$1 = \frac{1(1 + 1)}{2}$

$1 = 1$

Step 2. Assume true for k.

$1 + 2 + 3 + \dots k = \frac{k(k+1)}{2}$ Induction Hypothesis

Must show that the following is true

$1 + 2 + 3 + \dots k + k+1 = \frac{(k+1)(k+2)}{2}$

Proceed as follows:

Add k + 1 to both sides

$$\begin{aligned} 1 + 2 + 3 + \dots k + k+1 &= \frac{k(k+1)}{2} + k+1 \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{k^2 + 3k + 2}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

This is the statement we wished to establish for n = k+1. Therefore by PMI, S = N.

APPLICATIONS:

Prove by Principle of Mathematical Induction

1. $4 + 7 + 10 + \dots (3n+1) = \frac{n(3n+5)}{2}$
2. $1^2 + 2^2 + 3^2 + \dots n^2 = \frac{n(n+1)(2n+1)}{6}$
3. $1^3 + 2^3 + 3^3 + \dots n^3 = \frac{n^2(n+1)^2}{4}$
4. Sum of geometric series is given by the formula

$$S = \frac{a_1 - a_1 r^n}{1-r} \quad r \neq 1$$

SOLUTIONS TO APPLICATION PROBLEMS:

1. $4 + 7 + 10 + \dots (3n+1) = \frac{n(3n+5)}{2}$

a) Show $1 \in S_n$

$4 = \frac{1(3(1)+5)}{2}$

$4 = 4$

b) Assume true for n = k Induction Hypothesis.

$4+7+10+\ldots(3k+1) = \frac{k(3k+5)}{2}$

Show that this implies $4+7+10+\ldots 3K+1+3(K+1)+1 = \frac{(k+1)(3(k+1)+5)}{2}$

Add k+1 term to both sides.

$$4+7+10+\ldots 3k+1+3(k+1)+1 = \frac{k(3k+5)}{2} + 3(k+1)+1$$
$$= \frac{k(3k+5)+2(3(k+1)+1)}{2}$$
$$= \frac{3k^2+5k+6k+6+2}{2}$$
$$= \frac{3k^2+11k+8}{2}$$
$$= \frac{(3k+8)(k+1)}{2}$$

This is statement we wished to establish for n=k+1

∴ by PMI, S=N.

2. $1^2+2^2+3^2+\ldots n^2 = \frac{n(n+1)(2n+1)}{6}$

a) Show $1 \in S_n$

$\frac{1(1+1)(2+1)}{6}$

1 = 1

b) Assume $1^2+2^2+3^2+\ldots k^2 = \frac{k(k+1)(2k+1)}{6}$

must show that

$1^2+2^2+3^2+\ldots k^2+(k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

Add $(k+1)^2$ to both sides

$$1^2+2^2+3^2+\ldots k^2+(k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$
$$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$
$$= \frac{2k^3+3k^2+k+6k^2+12k+6}{6}$$
$$= \frac{2k^3+9k^2+13k+6}{6}$$
$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

∴ by PMI, S=N.

3. $1^3+2^3+3^3+\ldots n^3 = \frac{n^2(n+1)^2}{4}$

a) Show $1 \in S_n$

$$1^3 = \frac{1^2(1+1)^2}{4}$$

$$1 = 1$$

b) Assume true for n=k.

$$1^3+2^3+3^3+\ldots k^3 = \frac{k^2(k+1)^2}{4}$$

Show that this implies

$$1^3+2^3+3^3+k^3+(k+1)^3 = \frac{(k-1)^2(k+2)^2}{4}$$

Add $(k+1)^3$ to both sides

$$1^3+2^3+3^3+\ldots k^3+(k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$= \frac{k^2(k+1)^2+4(k+1)^3}{4}$$

$$= \frac{(k+1)^2\left[(k^2+4(k+1)\right]}{4}$$

$$= \frac{(k+1)^2\left[(k^2+4k+4)\right]}{4}$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

This is statement we wished to establish for n=K+1
Therefore by PMI, S=N.

4. $S = \frac{a_1 - a_1 r^n}{1-r} \quad r \neq 1$

1. Show $1 \in S_n$

$$a_1 = \frac{a_1 - a_1 r}{1-r}$$

$$= a_1\frac{(1-r)}{1-r}$$

$$a_1 = a_1$$

2. Assume true for n=k

$$a_1+a_1r+a_1r^2+\ldots a_1r^{k-1} = \frac{a_1-a_1r^k}{1-r}$$

Must show that this implies.

$$a_1+a_1r+\ldots a_1r^{k-1}+a_1r = \frac{a_1-a_1r^{k+1}}{1-r}$$

Proceed as follows: Add a_1r^k to both sides.

$$a_1+a_1r+a_1r^{k-1}+a_1r^k = \frac{a_1-a_1r}{1-r} + a_1r^k$$

$$= \frac{a_1-a_1r^k+(a_1r^k)(1-r)}{1-r}$$

$$= \frac{a_1-a_1r^k+a_1r^k-a_1r^{k+1}}{1-r}$$

$$= \frac{a_1-a_1r^{k+1}}{1-r}$$

This is statement we wished to establish for n=k+1.

SUMMARY:

1. Explain in your own words the steps necessary to show that a formula is true using the Principle of Mathematical Induction.

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